CPSC 270A, Spring 2003
Test 1 Solutions

1. Prove or disprove: $\log_2 n \in O(n^{1/2})$. **(10 points)**

   Answer

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log_2 n}{n^{1/2}} &= \lim_{n\to\infty} \frac{\frac{d}{dn}\log_2 n}{\frac{d}{dn}n^{1/2}}, \qquad \text{L'Hôspital's rule} \\
&= \lim_{n\to\infty} \frac{(\log_2 e)\frac{1}{n}}{\frac{1}{2\sqrt{n}}} \\
&= 2\log_2 e \lim_{n\to\infty} \frac{\sqrt{n}}{n} \\
&= 0.
\end{aligned}$$

   Thus, $\log_2 n \in O(n^{1/2})$.

2. Prove or disprove: $a^{\log_b c} = c^{\log_b a}$. **(10 points)**

   Answer

   Claim 1 $\log_a b^c = c \log_a b$.

   Proof Claim 1 Let $\log_a b = x$. Therefore, $b = a^x$. Then,

$$\log_a b^c = \log_a (a^x)^c = \log_a a^{xc} = xc = \log_a b \cdot c.$$

   $\square$ Claim 1

   Proof Let $\log_b a = x$. Therefore, $a = b^x$. Then, using Claim 1,

$$a^{\log_b c} = (b^x)^{\log_b c} = b^{x \log_b c} = b^{\log_b c^x} = c^x = c^{\log_b a}.$$

   $\square$

3. Consider the following recursive algorithm:

```
int function Q(n) {
    if (n == 1)
        return (1);
    else
        return (Q(n-1) + ((2 * n) - 1));
}
```

   (a) Set up a recurrence to count the number of additions and subtractions performed by this algorithm, given input $n > 0$. Solve the recurrence. **(10 points)**

   Answer Let $C(n)$ denote the number of additions and subtractions performed by the above algorithm when given an input $n > 0$. If the condition in the `if` statement is true then 0 additions and subtractions are performed; otherwise 1 addition and 1 subtraction is performed. Therefore,

$$C(n) = \begin{cases} 0, & \text{if } x = 1; \\ C(n-1) + 2, & \text{otherwise.} \end{cases}$$

Computing $C(n)$ for a few values we have,

$$\begin{aligned} C(1) &= 0, \\ C(2) &= C(1) + 2 = 2, \\ C(3) &= C(2) + 2 = 4, \\ C(4) &= C(3) + 2 = 6, \end{aligned}$$

leading us to conjecture that for all $n > 0, C(n) = 2n - 2$. The above computations provide us the base cases for $n < 5$.
Induction Hypothesis: For some k, $C(k) = 2k - 2$.
To show that: $C(k+1) = 2(k+1) - 2 = 2k$.
From the definition of C above and using the IH we have,

$$C(k+1) = C(k) + 2 = 2k - 2 + 2 = 2k.$$

$\square$

(b) Set up a recurrence for this function's value, for all $n > 0$, and solve it to determine what this algorithm computes. **(10 points)**

Answer Let $Q(n)$ denote the value of the function on input n. From the function we have,

$$Q(n) = \begin{cases} 1, & \text{if } n = 1; \\ Q(n-1) + 2n - 1, & \text{otherwise.} \end{cases}$$

Computing $Q(n)$ for a few values we have,

$$\begin{aligned} Q(1) &= 1, \\ Q(2) &= Q(1) + 2 \cdot 2 - 1 = 4, \\ Q(3) &= Q(2) + 2 \cdot 3 - 1 = 9, \end{aligned}$$

leading us to conjecture that for all $n > 0, Q(n) = n^2$. The above computations provide us the base cases for $n < 4$.
Induction Hypothesis: For some k, $Q(k) = k^2$.
To show that: $Q(k+1) = (k+1)^2$.
From the definition of Q above and using the IH we have,

$$Q(k+1) = Q(k) + 2(k+1) - 1 = k^2 + 2k + 1 = (k+1)^2.$$

$\square$

Note: In both the parts above, any identities you use should be proved as well. If in doubt, write down the identity that you want to use, and ask if it needs to be proved or not.

4. Solve the following recurrence relation:

$$R(x) = \begin{cases} 1, & \text{if } x = 0; \\ 2, & \text{if } x = 1; \\ 6R(x-1) - 9R(x-2) + 4, & \text{otherwise.} \end{cases}$$

(10 points)

Answer The above recurrence relation (RR) is an inhomogeneous RR. It's corresponding homogeneous RR is

$$R(x) - 6R(x-1) + 9R(x-2) = 0.$$

The characteristic equation for this homogeneous RR is

$$x^2 - 6x + 9 = 0$$

and has a repeated root at 3 since

$$x^2 - 6x + 9 = (x-3) \cdot (x-3).$$

Thus the general solution to the homogeneous RR is $\alpha 3^x + \beta x 3^x$.
Suppose $R(x) = c$ is a particular solution to the original inhomogeneous RR. Then, $c - 6c + 9c - 4 = 0$. Therefore, $c = 1$, i.e., $R(x) = 1$ is a particular solution to the inhomogeneous RR. Therefore, the general solution to the inhomogeneous RR is

$$R(x) = \alpha 3^x + \beta x 3^x + 1$$

Substituting the initial values for $x = 0$ and $x = 1$, we have

$$\begin{aligned} 1 &= \alpha + 1, \text{ and} \\ 2 &= 3\alpha + 3\beta + 1. \end{aligned}$$

Solving the simultaneous equations for α and β we have: $\alpha = 0$ and $\beta = 1/3$. Therefore,

$$R(x) = \frac{1}{3} x 3^x + 1.$$

5. Explain the essential idea behind Strassen's matrix multiplication algorithm. Thus derive clearly the number of integer multiplications involved in multiplying two $n \times n$ matrices. Assume $n = 2^k$ for some $k \geq 0$. **(10 points)**

Answer Strassen showed that two 2×2 integer matrices could be multiplied with performing only 7 integer multiplications instead of eight multiplications. His technique could be extended to multiply two $n \times n$ matrices ($n = 2^k$ for some $k > 0$) so that only 7 matrix multiplications, of $n/2 \times n/2$ matrices, are needed. Using this idea, the number of multiplications, denoted $M(n)$, performed using Strassen's matrix multiplication algorithm for multiplying two $2^k \times 2^k$ matrices is given by:

$$M(2^k) = \begin{cases} 7, & \text{if } k = 1; \\ 7M(2^{(k-1)}), & \text{otherwise.} \end{cases}$$

Computing $M(n)$ for some initial values we have

$$\begin{aligned} M(2^1) &= 7, \\ M(2^2) &= 7 \cdot 7 = 7^2, \\ M(2^3) &= 7 \cdot 7^2 = 7^3, \end{aligned}$$

leading us to conjecture that for all $k \geq 1, M(2^k) = 7^k$. This statement is easily proven using mathematical induction from the base cases above for $k < 4$ and from the definition of M above. Therefore, using the result in Question 2 above,

$$M(n) = M(2^k) = 7^k = 7^{\log_2 n} = n^{\log_2 7} < n^3.$$