CSE, Parallel Processing

EXISTENCE OF MAXIMAL SNAKES OF LENGTH $2(d+2)$

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Abstract

In this paper, the study of maximal induced paths (also known as maximal snakes) has been used as an approach to solve the problem of finding the longest snakes in hypercubes. In this connection, we show the existence of maximal d-snakes of length $2(d+2)$. It is a hope that this approach will result in finding longer snakes in hypercubes and possibly the longest one.

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1 Introduction

The problem of finding longest snakes in hypercubes was posed in 1958 (see [4]), and is still an open problem. Few characteristics of longest snakes are understood, but some recent observations of snakes have provided a possible route to predicting the length of longest snakes in higher-dimensional hypercubes.

In Section 2 we give an informal overview of snakes and hypercubes. In Section 3 we present some background information that leads to the present research. In Section 4 we present some relevant formal notation and definitions. In Section 5 we give an ouline of the proposed project and the work done till date. In Section 6 we summarize the significance and implications of our results.

2 Snakes in Hypercubes

Imagine a three-dimensional cube, which has eight corners and six faces, as an undirected graph, where the corners of the cube are the vertices of the graph and the lines where the faces meet are the edges of the graph. A *snake* is defined as a path, or a sequence of vertices in the graph, such that every pair of consecutive vertices is connected by an edge in the cube, while no two other vertices in the path are connected by an edge on the cube. Furthermore, this path must not double back on itself, that is, it must not form a cycle (loop). For example, in Figure 1, the path $c1, c2, c3$ is a snake of length 2 in the cube; the path $c1, c2, c3, c4$ is *not* a snake because the corners $c1$ and $c4$ are connected by an edge that is present in the cube; the path $c1, c2, c3, c7, c6, c2$ is *not* a snake because the path has a loop. The snake shown, $c1, c2, c3, c7, c8$ is a longest snake, of length 4, in the 3-dimensional cube.

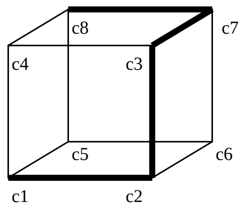

Figure 1: Longest Snake in a 3-dimensional Cube

This concept extends to hypercubes, which are "cubes" of dimensions greater than 3 (see Figure 2 for an example of a 4-dimensional hypercube). By referring to a snake in a hypercube, we are referring to the snake in the graphical representation of the hypercube. Alternatively, a snake in a hypercube is an *induced path* within the graph of the hypercube. (See [1] for a more formal definition.)

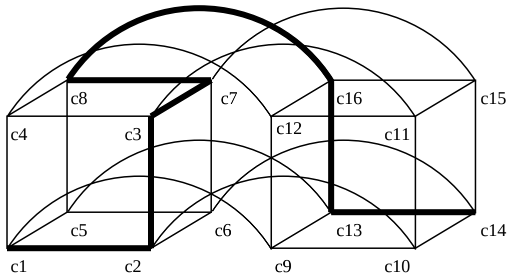

Figure 2: Longest Snake in a 4-dimensional Hypercube

The problem of finding a longest snake in a hypercube was proposed in the context of error-checking codes in 1958.(see [4]). The vertices of an n-dimensional hypercube can be uniquely coded in n bits, and a snake in this hypercube is a sequence of vertices that differ by exactly one bit. Selecting a suitable subset of the vertices of a snake gives a 1-bit error detecting code. Selecting a more constrained subset gives a 1-bit error correcting code. This problem also has some importance in connection with other applied problems such as electronic locking schemes and is still an open problem.

Analytical approaches have produced broad-ranging results for the longest snake problem. Exhaustive search is the only known method of finding a longest snake in a hypercube, and hence experimental approaches are uncommon because this method is computationally intractable in any realistic time frame. For instance, an exhaustive search for the longest snake in dimension 7 took approximately one month on a supercomputer dedicated to the task. Since exhaustive searches take an unacceptable amount of time, longest snakes are known only up through dimension 7. It is for this reason that we are continuing the analytical approach in an attempt to discover the properties of snakes.

3 Background

It was conjectured that there exists in every dimension d a maximal snake of every length from $2(d-1)$ up to the length of the longest snake in d. However, through an exhaustive generation of maximal 7-snakes, it was found that there exists no maximal 7-snake of length 13, thus disproving that conjecture in general. It is known that there exists a maximal snake of length $2d-1$ for dimensions up through 6, and that there exists no maximal snake of length $2d-1$ in any dimension higher than 6 (see [3]). Further, it is proven that for each $d > 1$, there is a maximal d-snake of length $2(d-1)$ (see [2]).

It is our hope that continued examination of maximal snakes of length $2d$, $2(d+1)$, $2(d+2)$, and so on, will provide a method of determining for what dimensions a certain length maximal snake exists.

4 Notation and Definitions

Here we present some additional notation, as well as formal definitions of some of the previously discussed concepts. This will serve as a foundation for the proofs that follow.

For each $d > 0$, we will denote the set $\{0, 1, \ldots, (d-1)\}$ by $\mathbf{d}$. $\mathcal{P}^{\mathbf{d}}$ will denote the set of all subsets of $\mathbf{d}$, and $\mathbf{2^d}$ will denote the set of characteristic functions of $\mathcal{P}^{\mathbf{d}}$. A *d-dimensional hypercubical graph* (or simply *d-dimensional hypercube*), denoted by H^d, is, by definition, an undirected graph with 2^d vertices, so that

1. each vertex is labeled with the characteristic function for a unique subset of $\mathbf{d}$, and
2. for any two vertices, v and w, with labels f_v and f_w respectively, (v, w) is an edge in H^d (i.e., v and w are neighbors in H^d) iff f_v and f_w differ in exactly one place.

Henceforth, we will not make any distinction between vertices in H^d and the labels assigned to them.

Then, by definition, a d-snake, S_d, is a sequence, $(s_0, s_1, \ldots, s_k)$, of elements of $\mathbf{2^d}$, such that for all i, j, $0 \leq i < j < k$, s_i and s_j are neighbors in H^d iff $j = i + 1$. The length of this snake is k, the number of edges of H^d in the snake.

Without loss of generality, we can assume that every d-snake $S_d = (s_0, s_1, \ldots, s_k)$, is such that $s_0 = \emptyset$. Then, we will represent S_d by a string $c_0 c_1 \ldots c_{k-1}$ over $\mathbf{d}$ so that for each $i, 0 \leq i \leq (k-1), s_i(c_i) \neq s_{(i+1)}(c_i)$, that is, the labels s_i and $s_{(i+1)}$ differ in the c_i-th place. Thus,

$$S_d = (s_0 = \emptyset, s_1, \ldots, s_k) = c_0 c_1 \ldots c_{k-1}.$$

Let $C_d = (c_1 c_2 \ldots c_k)$ be the sequence of symbols (dimensions) that define the snake S_d.

It is useful to introduce the following notation at this point.
Suppose $S_d = c_0, \ldots\ldots\ldots.c_{(k-1)}$ is a maximal d- snake. Then, $\bigcup_{0 \leq i < k} \{c_i\} = \mathbf{d}$ and for each $n \leq \mathbf{d}$

$$X^{(d,n)} = \{c | c \in \mathcal{P}^{\mathbf{d}} \text{ and } |c| = n\}$$

Let $S_d = (s_0, s_1, \ldots, s_k)$ be a snake.
S_d can be *extended backward* iff there exists a vertex c such that $(c, s_0, s_1, \ldots, s_k)$ is a snake.
S_d can be *extended forward* iff there exists a vertex e such that $(s_0, s_1, \ldots, s_k, e)$ is a snake.

A snake $S_d = (s_0, s_1, \ldots, s_k)$ is *maximal* in H^d iff, by definition, there is no vertex v in H^d such that $(v, s_0, \ldots, s_k)$ or $(s_0, \ldots, s_k, v)$ is a snake in H^d.

By *length* of a snake we will always mean the number of edges in that snake.

5 Existence of Maximal Snakes of length $2(d + 2)$

In our search to find the longest snake in a d-dimensional hypercube, which is one among the maximal snakes of various lengths in dimension d, we have found the existence of two patterns of maximal snakes of length $2(d + 2)$ for every dimension $d > 6$.

Lemma 1 *Suppose $S_d = (s_0 = \emptyset, s_1, \ldots, s_k)$ is a* **d***-snake. The snake cannot be extended backwards iff*

$$\bigcup_{s_i \in X^{(d,2)}} s_i = \{0, 1, 2, ..., (d-1)\} = \mathbf{d}.$$

Proof. Suppose, by way of contradiction, $e \in X^{(d,1)}$ such that $e = \{e_0\}$, and,

$$e, \emptyset, s_1, \ldots, s_k$$

is a snake. Since

$$\bigcup_{s_i \in X^{(d,2)}} s_i = \{0, 1, 2, ..., (d-1)\} = \mathbf{d},$$

there is a vertex $\{e_0, x\}$ in S_d. But, e is a neighbour of $\{e_0, x\}$ giving us a contradiction. □

Lemma 2 *For each $d > 6$, there is a maximal d-snake of length $2(d+2)$.*

Proof. For each $d > 6$, we will show that

$$01234203210540\,65...(d-1)(d-2)\,3(d-1)$$

is a maximal snake, thus proving the lemma. We have to prove two things.

1. the pattern above is a snake, and
2. the snake is a maximal d-snake.

The proof that the above pattern is a snakeis by induction. Once we prove that the pattern is a snake, to prove that the snake is a maximal snake we have to show that:

C1 it cannot be extended backward.

C2 it cannot be extended forward.

Base case $d = 7$

We need to show that the pattern

$$S_7 = 01234203210540\,65\,36$$

is a 7-snake, and that it is a maximal 7-snake.

As a sequence of vertices, S_7 is the sequence $s_0 = \emptyset, \ldots, s_{18}$ where

$$\begin{aligned}
s_1 &= \{0\} \\
s_2 &= \{0,1\} \\
s_3 &= \{0,1,2\} \\
s_4 &= \{0,1,2,3\} \\
s_5 &= \{0,1,2,3,4\} \\
s_6 &= \{0,1,3,4\} \\
s_7 &= \{1,3,4\} \\
s_8 &= \{1,4\} \\
s_9 &= \{1,2,4\} \\
s_{10} &= \{2,4\} \\
s_{11} &= \{0,2,4\} \\
s_{12} &= \{0,2,4,5\} \\
s_{13} &= \{0,2,5\} \\
s_{14} &= \{2,5\} \\
s_{15} &= \{2,5,6\} \\
s_{16} &= \{2,6\} \\
s_{17} &= \{2,3,6\} \\
s_{18} &= \{2,3\}.
\end{aligned}$$

It can be easily verified that for each i, $1 \leq i \leq 17$, the only neighbours of s_i are s_{i-1} and s_{i+1} and that s_0 and s_18 are not neighbours. Hence this is a snake. Now consider s_0 to s_{14} as the head, s_{15} and s_{16} as the body and s_{17} and s_{18} as the tail. s_0 to s_{14} are constant for all d-snakes for $d > 6$ and $\bigcup_{0 \leq i \leq 10} s_i = \{0,1,2,4,5\}$. The body consists of the vertex $\{2,6\}$ and the tail consists of the vertex $\{2,3\}$. Therefore,

$$\bigcup_{s \in X^{(d,2)} \cap S_7} s = \mathbf{d}.$$

Thus, using Lemma 1, the snake cannot be extended backward. The last vertex of the snake is $\{2,3\}$. Therefore, the snake can be extended forward, if possible, to one of the following vertices: $\{0,2,3\}$, $\{1,2,3\}$, $\{2,3,4\}$, $\{2,3,5\}$. But each of these vertices presents a problem in extending the snake forward as shown below.

1. $\{0,2,3\}$: $s_4 = \{0,1,2,3\}$ is a neighbour of this vertex.
2. $\{1,2,3\}$: $s_4 = \{0,1,2,3\}$ is a neighbour of this vertex .
3. $\{2,3,4\}$: $s_{10} = \{2,4\}$ is a neighbour of this vertex.
4. $\{2,3,5\}$: $s_{14} = \{2,5\}$ is a neighbour of this vertex.

Hence S_7 cannot be extended forward. Therefore, as both the conditions (C1) and (C2) are satisfied, S_7 is a maximal 7-snake of length $18 = 2(7+2)$.

Induction hypothesis For some $p > 6$, the p-snake

$$S_p = 01234203210540\,65........(p-1)(p-2)\,3(p-1) = (s_0 = \emptyset, s_1, \ldots, s_k)$$

satisfies conditions (C1) and (C2).

To show that

$$\begin{aligned} S_{p+1} &= 01234203210540\,65......(p+1-1)(p+1-2)\,3(p+1-1) \\ &= (u_0 = \emptyset, u_1, \ldots, u_{14}, u_{15}, \ldots, u_{2(p+1+1)}, u_{2(p+1+1)+1}, u_{2(p+1+2)}) \end{aligned}$$

is a maximal $(p+1)$-snake.

From the induction hypothesis,

1. u_0 to u_{14}, the head, is a snake, and
2. u_{15} to $u_{2(p+1+1-1)}$, the body is a snake.

The snake u_0 to $u_{2(p+1+1-1)}$ ends in the vertex $\{2, (p-1)\}$. Thus the two new vertices in the body of S_{p+1} are $\{2, (p-1), p\}$ and $\{2, 9\}$. Thus the head and the body of S_{p+1} is a snake.

The tail consists of the vertices $\{2, p, 3\}$ and $\{2, 3\}$. We now show that each of these can be used, in sequence, to extend the snake composed of the head and body of S_{p+1}.

$\{2, p, 3\}$**:** Any neighbour of this vertex must be in $X^{(d,2)}$ or $X^{(d,4)}$. The vertices in $X^{(d,4)}$ in the current snake are u_4,u_6,u_{12} and these vertices are clearly not neighbours of this vertex. The possible neighbours of this vertex that are also in $X^{(d,2)}$ are $\{2, p\}$,$\{2, 3\}$ and $\{3, p\}$. Among these, $\{2, p\}$ is the last vertex of the current snake, and $\{2, 3\}$ and $\{3, p\}$ do not appear in the current snake. Hence $\{2, p, 3\}$ can be used to extend the current snake.

$\{2, 3\}$**:** Any neighbour of this vertex must be in $X^{(d,1)}$ or $X^{(d,3)}$. The possible neighbours of this vertex that are also in $X^{(d,3)}$ are $\{0, 2, 3\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$, which do not appear in the current snake. The possible neighbours of this vertex that are also in $X^{(d,1)}$ are $\{2\}$ and $\{3\}$ which do not appear in the current snake. Hence $\{2, 3\}$ can be used to extend the current snake.

Hence proved that S_{p+1} is a snake.

Now we show that S_{p+1} is a maximal snake.

$$S_{p+1} = 01234203210540\,65......(p+1-1)(p+1-2)\,3(p+1-1)$$

From the above description of the vertices of S_{p+1}, it can be verified that

$$\bigcup_{s \in X^{(d,2)} \cap S_{(p+1)}} s = (\mathbf{p+1})$$

Thus by Lemma 1, condition (C1) is satisfied.

The last vertex of S_{p+1} is $\{2,3\}$. Therefore, the snake can be extended forward, if possible, to one of the following vertices: $\{0,2,3\}$, $\{1,2,3\}$, $\{2,3,4\}$,$\{2,3,5\}$,......... $\{2,3,p\}$. But each of these vertices presents a problem in extending the snake forward as shown below.

1. $\{2,3,0\}$: $u_4 = \{0,1,2,3\}$ is a neighbour of this vertex.
2. $\{2,3,1\}$: $u_4 = \{0,1,2,3\}$ is a neighbour of this vertex .
3. $\{2,3,4\}$: $u_{10} = \{4,2\}$ is a neighbour of this vertex.
4. $\{2,3,5\}$: $u_{14} = \{2,5\}$ is a neighbour of this vertex.
5. $\{2,3,p\}$: $u_{2(p+1+1)+1}$ is the same as this vertex.
6. For each $x, 6 \leq x(p+1-2)$, there is a vertex $\{2,x\}$ in the current snake. Hence $\{2,3,6\}$......$\{2,3,(p+1-2)\}$ have neighbours in the body.

Hence S_{p+1} satisfies condition (C2).

As both the conditions (C1) and (C2) are met, S_{p+1} is a maximal snake. Thus for each $d > 6$, there is a maximal d-snake of length $2(d+2)$. $\square$

Lemma 3 01234203210540 35 *is a maximal* 6*-snake of length* $16 = 2(6+2)$.

PROOF.
01234203210540 is the head of this snake as well as that of S_7 used in the proof of Lemma 2. Hence this is a snake.
As a sequence of vertices, S_6 is the sequence $s_0 = \emptyset, \ldots, s_{16}$ where

$$\begin{aligned}
s_1 &= \{0\} \\
s_2 &= \{0,1\} \\
s_3 &= \{0,1,2\} \\
s_4 &= \{0,1,2,3\} \\
s_5 &= \{0,1,2,3,4\} \\
s_6 &= \{0,1,3,4\} \\
s_7 &= \{1,3,4\} \\
s_8 &= \{1,4\}
\end{aligned}$$

$$
\begin{aligned}
s_9 &= \{1,2,4\} \\
s_{10} &= \{2,4\} \\
s_{11} &= \{0,2,4\} \\
s_{12} &= \{0,2,4,5\} \\
s_{13} &= \{0,2,5\} \\
s_{14} &= \{2,5\} \\
s_{15} &= \{2,3,5\} \\
s_{16} &= \{2,3\}
\end{aligned}
$$

The tail consists of the vertices $\{2,3,5\}$ and $\{2,3\}$. we now show that each of these, can be used, in sequence, to extend the snake composed of the head in S_6.

$\{2,5,3\}$: Any neighbour of this vertex must be in $X^{(d,2)}$ or $X^{(d,4)}$. The vertices in $X^{(d,4)}$ in the current snake are s_4,s_6,s_{12} and these vertices are clearly not neighbours of this vertex. The possible neighbours of this vertex that are also in $X^{(d,2)}$ are $\{2,3\}$,$\{2,5\}$,$\{5,3\}$. Among these vertices,$\{2,5\}$ is the last vertex in the current snake and $\{2,3\}$ and $\{5,3\}$ do not appear in the current snake. Hence $\{2,5,3\}$ can be used to extend the current snake.

$\{2,3\}$: Any neighbour of this vertex must be in $X^{(d,1)}$ or $X^{(d,3)}$. The possible neighbours of this vertex that are also in $X^{(d,3)}$ are $\{0,2,3\}$, $\{1,2,3\}$, $\{2,3,4\}$ and $\{2,3,5\}$, which do not appear in the current snake. The possible neighbours of this vertex that are also in $X^{(d,1)}$ are $\{2\}$ and $\{3\}$ which do not appear in the current snake. Hence $\{2,3\}$ can be used to extend the current snake.

Hence proved that S_6 is a snake.

$\bigcup_{s\in X^{(d,2)}\cap S_6} s = \{0,1,2,3,4,5\}$. Therefore, S_6 cannot be extended backwards, thus satisfying (C1).

Note that the head of S_6 and that of $S_(p+1)$ used in the proof of Lemma 2 are the same. Also note that both these snakes end in the vertex $\{2,3\}$. It can be verified that the argument presented in Lemma 2 for why $S_(p+1)$cannot be extended forward applies here and thus S_6 cannot be extended forward satisfying (C2).
As both the conditions (C1) and (C2) are met, s_6 is a maximal snake. □

Lemmas 2 and 3 together establish that

Theorem 1 *For each $d \geq 6$,there is a maximal d-snake of length $2(d+2)$.* □

For $d \geq 6$, the same proof technique is also applicable to the pattern

$$0123104312 \quad 54.........(d-2)(d-3) \quad 3(d-1)2(d-2)32$$

of length $2(d+2)$, where 0123104312 is the head, $54...(d-2)(d-3)$ is the body, and $3(d-1)2(d-2)32$ is the tail.

6 Conclusions

To solve the problem of longest snakes in hypercubes, we must be able to find the longest snakes, or at least determine the length of the longest snakes, in any given dimensional hypercube. We now have knowledge about the existence of length $2(d+2)$ maximal snakes. (In another paper coming out of this project, it is shown that for $d \geq 5$, there exists a family of patterns of length $2(d+1)$ maximal snakes. Our hope is to continue this search so as to find the limit i, for which a pattern extends as a maximal snake of length $2(d+i)$ for every dimension d. It is our hope that, by examining the methods of proof that were used in this paper and by extending our study to patterns of length $2(d+3)$, $2(d+4)$......, we may discover a general proof technique that can be applied to maximal snakes of any length. The ability to determine the existence of any length maximal snake in any dimension would be a significant step in the search for the length of longest snakes in hypercubes.

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