A Characterisation of Optimal Channel Assignments for Cellular and Square Grid Wireless Networks

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Abstract

In this paper we first present a uniformity property that characterises optimal channel assignments for networks arranged as cellular or square grids. Then, we present optimal channel assignments for cellular and square grids; these assignments exhibit a high value for δ_1 — the separation between channels assigned to adjacent stations. We prove an upper bound on δ_1 for such optimal channel assignments. This upper bound is greater than the value of δ_1 exhibited by our assignments. Based on empirical evidence, we conjecture that the value our assignments exhibit is a tight upper bound on δ_1.

1 Introduction

The enormous growth of wireless networks has made the efficient use of the scarce radio spectrum important. A "Frequency Assignment Problem" (FAP) models the task of assigning frequencies (channels) from a radio spectrum to a set of transmitters and receivers, satisfying certain constraints [9]. The main difficulty in an efficient use of the radio spectrum is the *interference* caused by unconstrained simultaneous transmissions. Interferences can be eliminated (or at least reduced) by means of suitable *channel assignment* techniques, which partition the given radio spectrum into a set of disjoint channels that can be used simultaneously by the stations while maintaining acceptable radio signals. Since radio signals get attenuated over distance, two stations in a network can use the same channel without interferences provided the stations are spaced sufficiently apart. The minimum distance at which channels can be reused with no interferences is called the *co-channel reuse distance* (or simply *reuse distance*) and is denoted by σ.

In a *dense* network – a network where there are a large number of transmitters and receivers in a small area – interference is more likely. Thus, reuse distance needs to be high in such networks. Moreover, channels assigned to nearby stations must be separated in value by at least a gap which is inversely proportional to the distance between the two stations. A minimum *channel separation* δ_i is required between channels assigned to stations at distance i, with $i < \sigma$, such that δ_i decreases when i increases

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[8]. σ is said to place the *co-channel reuse distance constraint*, and the vector $\vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ is said to place the *channel separation constraint* on the channel assignment problem.

The purpose of channel assignment algorithms is to assign channels to transmitters in such a way that (1) the co-channel reuse distance and the channel separation constraints are satisfied, and (2) the *span* of the assignment, defined to be the difference between the highest and the lowest channels assigned, is as small as possible [2].

This paper has two significant contributions:

1. A characterisation of optimal channel assignments for cellular and square grids. We essentially show a nice uniformity across the grid that every optimal assignment must satisfy. (See Section 3.)

2. Optimal channel assignments for cellular and square grids where the channel separation between adjacent stations is large. We prove an upper bound on δ_1 for such optimal channel assignments. This upper bound is greater than the value of δ_1 exhibited by our assignments. Based on empirical evidence, we conjecture that the value our assignments exhibit is a tight upper bound on δ_1. (See Section 4.)

In Section 2 we formally define the problem of channel assignments and its formulation as a colouring problem, and provide a brief literature survey. We also outline the general strategy we use for our optimal colourings discussed in Section 4. In Section 3 we first define the cellular (Section 3.1) and square (Section 3.2) grids, and point out some useful properties of these grids. Then, in Section 3.3, we prove a characterisation of optimal colourings for cellular and square grids. In Section 4 we present our colourings and prove that they are optimal. Then, in Section 5, we present an upper bound on the value of the channel separation among adjacent stations as witnessed by optimal colourings.

2 Preliminaries

Formally, the *Channel Assignment Problem with Separation* (CAPS) can be modelled as an appropriate colouring problem on an undirected graph $G = (V, E)$ representing the network topology, whose vertices in V correspond to stations, and edges in E correspond to pairs of stations that can hear each other's transmission [2]. The colour assigned to a particular vertex corresponds to the frequency channel assigned to the corresponding station. For a graph G, we will denote the distance between any two vertices in the graph, i.e., the number of edges in a shortest path between the two vertices, by $d_G(\cdot,\cdot)$. (When the context is clear, we will denote the distance as simply $d(\cdot,\cdot)$.) CAPS is then defined as:

> **CAPS** $(G, \sigma, \vec{\delta})$
> Given an undirected graph G, an integer $\sigma > 1$, and a vector of positive integers $\vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$, find an integer $g > 0$ so that there is a function $f : V \to \{0, \ldots, g\}$, such that for all $u, v \in G$, for each i, $1 \le i \le \sigma - 1$, if $d(u, v) = i$, then $|f(u) - f(v)| \ge \delta_i$.

This assignment is referred to as a g-$L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ colouring of the graph G [7], and **CAPS** $(G, \sigma, \vec{\delta})$ is sometimes referred to as the $L(\vec{\delta})$ colouring problem for G. Note that a g-$L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ uses only the $(g+1)$ colours in the set $\{0, \ldots, g\}$, but does *not* necessarily use all the $(g+1)$ colours. A g-$L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ colouring of G is *optimal* iff g is the smallest number witnessing a solution for **CAPS** $(G, \sigma, \vec{\delta})$.

Finding the optimal colouring for general graphs has been shown to be NP-complete. The problem remains NP-complete even if the input graphs are restricted to planar graphs, bipartite graphs, chordal graphs, and split graphs [4]. Most of the work on this problem has dealt with specific graphs such as grids and rings, for small reuse distance (σ) values, and for small channel separation (δ_i) values, e.g., optimal $L(1,1)$ colourings for rings and bidimensional grids [1], optimal $L(2,1)$ and $L(2,1,1)$ colourings for hexagonal, bidimensional, and cellular grids [2], etc. Recently, Bertossi et al [3] exhibited optimal

$L(\delta_1, 1, \ldots, 1)$ colourings, for $\delta_1 \leq \lfloor \sigma/2 \rfloor$, for bidimensional grids and rings. (See [3] for a succinct literature survey of this problem.) Below, we refer to $L(\cdot, 1, \ldots, 1)$ colourings by $L(\cdot, \vec{1}_k)$ colourings.

As pointed out in [2], a lower bound for the $L(1, \vec{1}_k)$ colouring problem is also a lower bound for the $L(\delta, \vec{1}_k)$, $\delta > 1$. Given an instance of **CAPS**, consider the augmented graph obtained from G by adding edges between all those pairs of vertices that are at a distance of at most $\sigma - 1$. Clearly, then, the size (number of vertices) of any clique in this augmented graph places a lower bound on an $L(1, \vec{1}_{\sigma-1})$ colouring for G; the best such lower bound is given by the size of a maximum clique in the augmented graph.

In each graph, G, for each σ, we identify a canonical sub-graph, $T(G, \sigma)$, of the graph so that the vertices of $T(G, \sigma)$ induce a clique in the augmented graph of the graph. We will refer to $T(G, \sigma)$ as a *tile*. When the context is clear, we will refer to the size of $T(G, \sigma)$ simply as $c(\sigma)$.

Most (but not all) of the assignment schemes described in this paper follow the pattern: for a given graph G, and for a given σ, (1) identify $T(G, \sigma)$, (2) find the number of vertices in $T(G, \sigma)$, and hence a lower bound for the given assignment problem, (3) describe a colouring scheme to colour all the vertices of $T(G, \sigma)$, (4) demonstrate a tiling of the entire graph made up of $T(G, \sigma)$ to show that the colouring scheme described colours the entire graph, and (5) show that the colouring scheme satisfies the given reuse distance and channel separation constraints.

3 A Characterisation of Optimal Colourings

We first introduce the conventions we follow to represent square grids and cellular grids. We explain tilings in both grids, and define some notation. Then we present our characterisation of optimal colourings in cellular and square grids.

For any d-dimensional lattice, $\mathcal{L}$, the minimal distance in the lattice is denoted by $\mu(\mathcal{L})$. The infinite graph, denoted $\mathcal{G}(\mathcal{L})$, corresponding to the lattice $\mathcal{L}$ consists of the set of lattice points as vertices; each pair of lattice points that are at a distance $\mu(\mathcal{L})$ constitute the edges of $\mathcal{G}(\mathcal{L})$.

The lattice $\mathbf{Z}^d$ is the set of ordered d-tuples of integers, and $\mathbf{A}_d$ is the hyperplane that is a subset of $\mathbf{Z}^{d+1}$, and is characterised as the set of points in $\mathbf{Z}^{d+1}$ such that the coordinates of each point add up to zero. $\mu(\mathbf{Z}^d) = 1$, and the minimal length vectors in $\mathbf{Z}^d$ are the unit vectors in each dimension. For each $d > 0$, for each $i, j, 0 \leq i, j \leq d, i \neq j$, define $\lambda_{ij}^d = (x_0, \ldots, x_d)$ where $x_i = 1$, $x_j = -1$, and for each $k, 0 \leq k \leq d, k \neq i, j, x_k = 0$. Then, $\mu(\mathbf{A}_d) = \sqrt{2}$, and the set of minimal length vectors in $\mathbf{A}_d$ is $\{\lambda_{ij}^d \mid i, j, 0 \leq i, j \leq d, i \neq j\}$. (See [5, 10] for more on these lattices.)

The infinite 2-dimensional square grid is, then, $\mathcal{G}(\mathbf{Z}^2)$, and the infinite 2-dimensional cellular grid is $\mathcal{G}(\mathbf{A}_2)$.

3.1 Cellular Grids

For a given value of σ, two kinds of tiles can be identified: *triangular* and *hexagonal*. The tiles are shown in Figure 1(a).

It can be easily shown that:

Lemma 1. 1. *The number of vertices in a* triangular tile *corresponding to reuse distance σ, denoted by $c_T(\sigma)$, is given by $c_T(\sigma) = \frac{\sigma(\sigma+1)}{2}$.*

2. *The number of vertices in a* hexagonal tile *corresponding to reuse distance σ, denoted by $c_H(\sigma)$, is given by $c_H(\sigma) = (3\sigma^2 + (\sigma \bmod 2))/4$. Thus, when $\sigma = 2k+1$ (odd), $c(\sigma) = 3k^2 + 3k + 1$.* □

From the above Lemma, we observe that for a given value of σ, the size of a *hexagonal tile* is greater than the size of a *triangular tile*. As mentioned in the previous section, the size of the maximum clique *i.e.* $c_H(\sigma)$ places a lower bound on the colouring of $\mathcal{G}(\mathbf{A}_2)$. Henceforth, for $\mathcal{G}(\mathbf{A}_2)$, we consider only *hexagonal tiles* and the word *tile* refers to *hexagonal tile* unless otherwise mentioned. Also, we refer to $c_H(\sigma)$ simply as $c(\sigma)$.

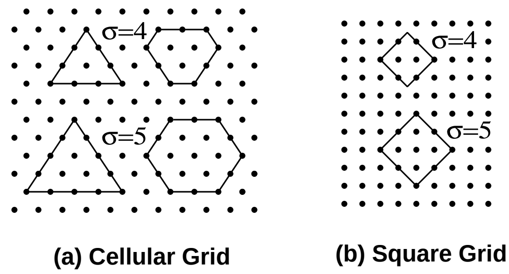

Figure 1: Cliques in Cellular and Square Grids

For a particular σ, hexagons are regular with sides of $\lceil \frac{\sigma}{2} \rceil$ vertices if σ is odd. In case σ is even, alternate sides of the hexagon are equal and consecutive sides have $\frac{\sigma}{2}$ and $\lceil \frac{\sigma+1}{2} \rceil$ vertices respectively.

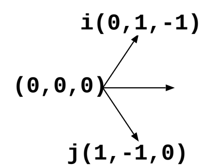

Figure 2: Basis vectors in $\mathbf{A}_2$

Figure 2 shows the coordinate system we use for representing vertices in $\mathbf{A}_2$ where $(0, 1, -1)$ and $(1, -1, 0)$ indicate the basis vectors i and j. In the table in Figure 3, we list the coordinates of the corners of a hexagon in clockwise order, for various values of σ. We start with the left-most vertex, which we refer to as the *origin* and assign $(0, 0)$ as its coordinates. Consider the arrangement of tiles as shown in Figure 4. It is clear that such an arrangement will tile all of $\mathbf{A}_2$. Note that any translation of this tiling will also tile $\mathbf{A}_2$.

In such a tiling, we will refer to two tiles as neighbours if there is an edge, e_1, of one and an edge, e_2, of the other such that at least two points on e_1 have neighbours on e_2 and vice versa. In such a tiling of $\mathbf{A}_2$, every tile is surrounded by six neighbouring tiles. For any tile H, we will refer to the neighbouring tiles as $H_0, H_1, \ldots, H_5$ as shown in Figure 4. If the coordinates of the origin of H are $(0, 0)$, then the origins of $H_0, H_1, \ldots, H_5$ will have coordinates $(-\lfloor \frac{\sigma}{2} \rfloor, -\sigma)$, $(\lceil \frac{\sigma}{2} \rceil, -\lfloor \frac{\sigma}{2} \rfloor)$, $(\sigma, \lceil \frac{\sigma}{2} \rceil)$, $(\lfloor \frac{\sigma}{2} \rfloor, \sigma)$, $(-\lceil \frac{\sigma}{2} \rceil, \lfloor \frac{\sigma}{2} \rfloor)$ and $(-\sigma, -\lceil \frac{\sigma}{2} \rceil)$ respectively. These points are marked in Figure 4. The edge of tile H which is adjacent to the tile H_i will be denoted by t_i.

Definition 1. *In a cellular grid tile, we define a* diagonal *to be a line formed by all vertices having the same i^{th} coordinate. In a tile with origin (i_o, j_o), a diagonal corresponding to the coordinate i_c is*

σ mod $4 = 0$	σ mod $4 = 2$	σ mod $4 = 1, 3$
$(0, 0)$	$(0, 0)$	$(0, 0)$
$(\frac{\sigma}{2} - 1, 0)$	$(\frac{\sigma}{2}, 0)$	$(\lfloor \frac{\sigma}{2} \rfloor, 0)$
$(\sigma - 1, \frac{\sigma}{2})$	$(\sigma - 1, \frac{\sigma}{2} - 1)$	$(\sigma - 1, \lfloor \frac{\sigma}{2} \rfloor)$
$(\sigma - 1, \sigma - 1)$	$(\sigma - 1, \sigma - 1)$	$(\sigma - 1, \sigma - 1)$
$(\frac{\sigma}{2} - 1, \sigma - 1)$	$(\frac{\sigma}{2}, \sigma - 1)$	$(\lfloor \frac{\sigma}{2} \rfloor, \sigma - 1)$
$(0, \frac{\sigma}{2})$	$(0, \frac{\sigma}{2} - 1)$	$(0, \lfloor \frac{\sigma}{2} \rfloor)$

Figure 3: Coordinates of the corners of a Hexagonal Tile

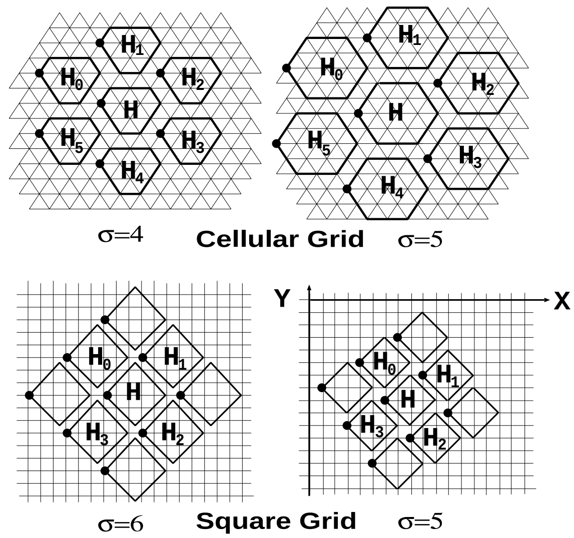

Figure 4: Tiling of $\mathbf{A}_2$ and $\mathbf{Z}^2$

represented as $L_{i_c - i_o}$ *and* $(i_c - i_o)$ *is called the* diagonal number.

In a tile corresponding to reuse distance σ, there are σ *diagonals*. Figure 5 shows the *diagonals* in a cellular grid.

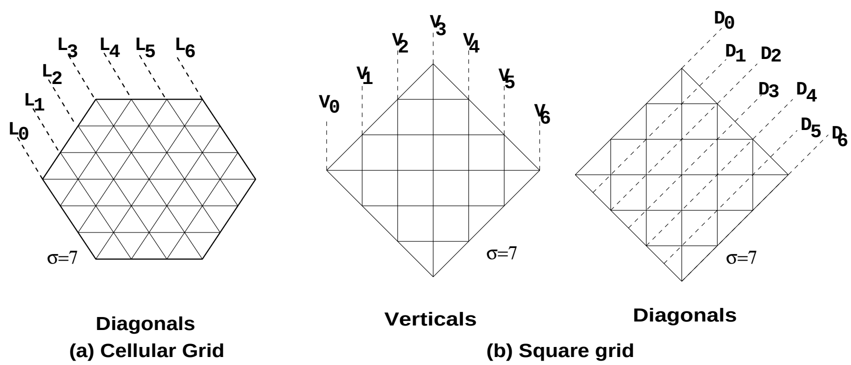

Figure 5: Verticals and Diagonals

3.2 Square Grids

As mentioned in Section 2, the size of the maximum clique for a particular σ places a lower bound on the colouring of $\mathcal{G}(\mathbf{Z}^2)$. The following Lemma gives a formula for the size of such a clique which is also referred to as a tile.

Lemma 2. *The number of vertices in a tile corresponding to reuse distance* σ*, denoted by* $c(\sigma)$ *is given by* $c(\sigma) = \lceil \frac{\sigma^2}{2} \rceil$*. Thus, when* $\sigma = 2k+1$ *(odd),* $c(\sigma) = 2k^2 + 2k + 1$*.* □

For a particular σ, tiles are diamonds with their diagonals along the X and Y axes (as shown in Figure 1(b)) and every side contains $\lceil \frac{\sigma}{2} \rceil$ vertices. They tile the entire grid $\mathbf{Z}^2$. In the case of odd σ, every corner of the tile corresponds to a vertex on the grid. We use the vectors $(1,0)$ and $(0,1)$ as the basis vectors i and j for representing points in $\mathbf{Z}^2$. Then, the coordinates of the vertices of the corners of a tile (in clockwise order, starting with the left-most vertex) are $(0,0)$, $(\lfloor\frac{\sigma}{2}\rfloor, \lfloor\frac{\sigma}{2}\rfloor)$, $(\sigma-1,0)$ and $(\lfloor\frac{\sigma}{2}\rfloor, -\lfloor\frac{\sigma}{2}\rfloor)$. In the case of even σ, only opposite corners of the tile along the X direction correspond to vertices on the grid, their coordinates being $(0,0)$ and $(\sigma-1,0)$.

Consider the arrangement of tiles as shown in Figure 4. It is clear that such an arrangement will tile all of $\mathbf{Z}^2$. Note that any translation of this tiling will also tile $\mathbf{Z}^2$.

In such a tiling, we will refer to two tiles as neighbours if there is an edge, e_1, of one and an edge, e_2, of the other such that at least two points on e_1 have neighbours on e_2 and vice versa. In such a tiling of $\mathbf{Z}^2$, every tile is surrounded by four neighbouring tiles. For any tile H, we will refer to the neighbouring tiles as H_0, H_1, H_2 and H_3. If the coordinates of the left-most vertex of H (which we refer to as the *origin*) are $(0,0)$, then the origins of H_0, H_1, H_2 and H_3 will have coordinates $(-\lfloor\frac{\sigma}{2}\rfloor, \lceil\frac{\sigma}{2}\rceil)$, $(\lceil\frac{\sigma}{2}\rceil, \lfloor\frac{\sigma}{2}\rfloor)$, $(\lfloor\frac{\sigma}{2}\rfloor, -\lceil\frac{\sigma}{2}\rceil)$ and $(-\lceil\frac{\sigma}{2}\rceil, -\lfloor\frac{\sigma}{2}\rfloor)$ respectively. These points are shown in Figure 4. The edge of tile H which is adjacent to the tile H_i will be denoted by t_i.

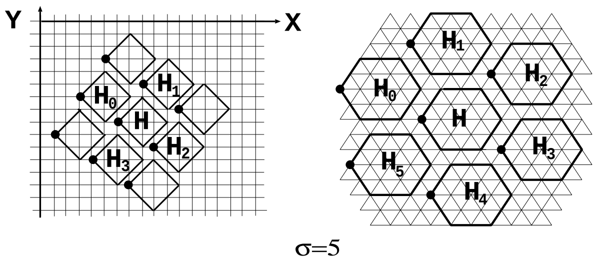

Figure 6: Possible tiling of $\mathbf{Z}^2$ and $\mathbf{A}_2$ for odd σ

There is another kind of tiling possible in both cellular and square grids for odd reuse distances, as shown in Figure 6 for $\sigma = 5$. We shall refer to the tiling shown in Figure 4 as *Tiling A* and the one in Figure 6 as *Tiling B*.

Definition 2. *In a square grid tile, we define a* vertical *to be a line formed by all vertices having the same X-coordinate. In a tile with origin (i_o, j_o), a vertical corresponding to the coordinate i_c is represented as $V_{i_c - i_o}$ and $(i_c - i_o)$ is called the* vertical number*.*

Definition 3. *In a square grid tile, we define a* diagonal *to be a line of the form $i - j = c$ where c is a constant. It is represented as D_i where i, called the* diagonal number *is given by $(i-j) \bmod \sigma$.*

In a tile corresponding to reuse distance σ, there are σ *diagonals/verticals* as the case may be. Figure 5 shows *verticals* and *diagonals* of a square grid tile.

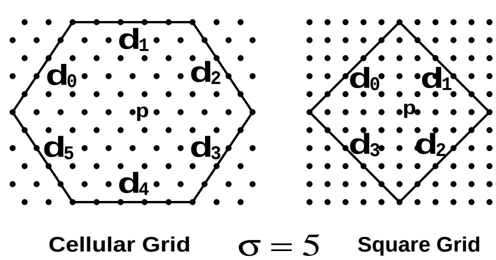

Figure 7: *Bounding Box* $B(p)$ with edges marked

Definition 4.

1. *Consider a point p in a square/cellular grid and consider all points which are at a distance σ from p, where σ is the reuse distance. In the case of square grids, all these points form a diamond centered at p and in the case of cellular grids, they form a hexagon centered at p. This diamond/hexagon will be called the* bounding box surrounding point p *and will be denoted by $B(p)$. The edges, considered in a clockwise direction, are denoted by $d_0, d_1, \ldots, d_3$ in case of square grids and $d_0, d_1, \ldots, d_5$ in case of cellular grids, as shown in Figure 7.*

2. *Consider the* bounding box *for point p. Every edge contains $\sigma - 1$ vertices apart from the two corners. Each corner, which belongs to two edges $d_{\bar{i}}$ and $d_{\overline{i+1}}$, is taken to be a part of the second edge $d_{\overline{i+1}}$, where $\bar{i}$ refers to $(i \bmod s)$, s being 4 in case of square grids and 6 in case of cellular grids. For each edge, we number the vertices consecutively, clockwise, starting with 0 being assigned to the left-corner vertex. These numbers are called* position numbers. *This is shown in Figures 8, 9, 10 and 11.*

3.3 Optimal Colouring Schemes

A colouring scheme is optimal if it uses the smallest possible number of colours. In other words, a colouring which uses colours from the set $\{0, 1, \ldots, g\}$ will be optimal if it uses the smallest possible value for g. From Lemma 1 and Lemma 2, we know that $c(\sigma)$ is a lower bound on the number of colours used. We are concerned only with such colouring schemes which use exactly $c(\sigma)$ different colours.

We already know that σ is the *minimum* distance at which channels can be reused. In other words, the same colour can be used for vertices which are at distance σ or greater. The following lemma establishes that in an optimal colouring the nearest vertex where a colour is reused is no more than distance σ away.

Lemma 3. *Consider an optimal colouring scheme for a wireless network modelled as an infinite square or cellular grid with reuse distance σ. For a given point p, there exists at least one point at distance σ from p which has the same colour as p.*

Proof. Let us assume that, on the contrary, there is no point at distance σ from p which has the same colour as p. Thus, no point inside, or on the boundary of, $B(p)$ is assigned the same colour as that of p.

Now, consider one of the edges of $B(p)$, say d_0 and a tile inside $B(p)$ such that one of its edges t_0 is completely contained in this edge of $B(p)$. Clearly, p is not in this tile. Since we have an optimal colouring, one of the points in the tile must be assigned the same colour as the colour assigned to p. This is a contradiction, and hence the result. □

We now present a theorem using which we will be able to establish an important property of optimal colouring schemes.

Theorem 1. *Consider an optimal colouring scheme for a wireless network modelled as an infinite square or cellular grid with reuse distance σ. For every point p, there is a position number n, such that each point corresponding to this position number on each edge of the bounding box surrounding p has the same colour as p. Moreover, $n = \lceil \frac{\sigma}{2} \rceil - 1$ or $n = \lceil \frac{\sigma}{2} \rceil$*

Proof. Consider the edge d_0 of the bounding box around p, $B(p)$. Consider the k different tiles, each of whose edge t_0 is a part of the edge d_0 of $B(p)$, where $\sigma = 2k + 1$ for odd σ and $\sigma = 2k$ for even σ. Refer to Figures 8, 9, 10, 11.

Let $P(i)$ denote the sequence of position numbers on d_0 of $B(p)$ that are on the edge t_0 of the i-th of these k tiles. In the case of odd σ, $P(i)$ are given by:

$$\begin{aligned} P(1) &= \langle 1, \ldots, k+1 \rangle \\ P(2) &= \langle 2, 3, \ldots, k+2 \rangle \end{aligned}$$

$$\vdots \quad \vdots$$

$$P(k) = \langle 2+(k-2), 2+(k-1), \ldots, 2k \rangle \tag{1}$$

In case σ is even, $P(i)$ are given by:

$$\begin{aligned} P(1) &= \langle 1, 2, \ldots, k \rangle \\ P(2) &= \langle 2, 3, \ldots, k+1 \rangle \\ &\vdots \quad \vdots \\ P(k) &= \langle 2+(k-2), 2+(k-1), \ldots, 2k-1 \rangle \end{aligned} \tag{2}$$

Since the colouring is optimal, the colour c, that the point p is coloured in, must appear somewhere on each of these tiles. Except for the edge t_0, each of these tiles is completely contained within $B(p)$. Thus, the colour c must appear on the edge t_0 of each of these tiles, otherwise, the reuse constraint is violated. Since no pair of vertices with position numbers 1, 2, …, $2k$ on the edge d_0 of $B(p)$ are at a distance σ, it must be the case that the colour c is assigned to some vertex that is common to all the above tiles. In case of odd σ, as seen from Equation (1), the only two common vertices are the ones with position numbers k and $k+1$, and hence, one of these two vertices must be assigned the colour c. In case of even σ, we see from Equation (2) that the only common vertex is the one corresponding to position number k, and hence, it has to be assigned the colour c. A similar argument establishes that on each edge of $B(p)$, the vertices corresponding to position numbers k and $k+1$ in case of odd σ and k in case of even σ are the only possible candidates for being assigned colour c.

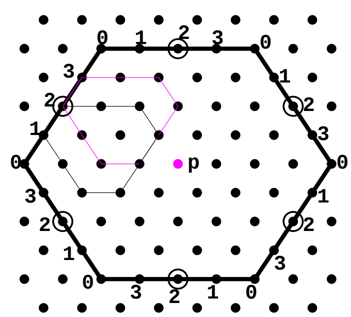

Figure 8: Cellular Grid *Bounding Box* for $\sigma = 4$ with position numbers

Now, in case of odd σ, let q be the vertex, corresponding to position number k on the edge d_0 of $B(p)$, that is assigned colour c (see Figures 9 and 11). Suppose, by way of contradiction, the vertex with position number $k+1$ on the edge d_1 of $B(p)$ is assigned colour c. Let us name this vertex x. We will now consider cellular and square grids separately in two different cases.

Case 1 (Cellular Grids): Consider the bounding box $B(q)$. The edge d_2 of $B(q)$ passes through the vertex with position number k on the edge d_1 of $B(p)$ as shown in Figure 9. By the above argument, one of the two vertices with position numbers k or $k+1$ on this edge d_2 of $B(q)$ must be assigned colour c. But both these vertices are at a distance less than σ from the vertex x. Therefore, x cannot be assigned colour c, implying that the vertex with position number k on the edge d_1 must be assigned colour c.

Case 2 (Square Grids): Consider the bounding boxes $B(q)$ and $B(x)$. Let r be the point of intersection of the edges d_1 of $B(q)$ and d_0 of $B(x)$ (see Figure 11). If both q and x are coloured c, it follows that r should be assigned the colour c. This is not possible because r lies within the bounding box $B(p)$ of point p which is also coloured c. This implies that the vertex with position number k on the edge d_1 must be assigned colour c.

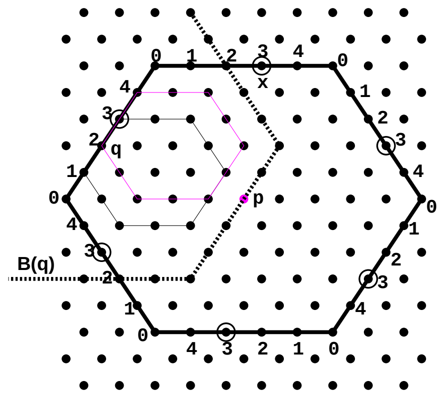

Figure 9: Cellular Grid *Bounding Box* for $\sigma = 5$ with position numbers

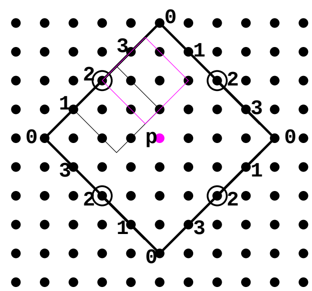

Figure 10: Square Grid *Bounding Box* for $\sigma = 4$ with position numbers

Similar arguments in both cases above establish that, if the vertex with position number k on any one edge of $B(p)$ is coloured the same as the colour of p, then on each edge of $B(p)$, the vertex with position number k is also coloured the same as p. □

The following characterisation of optimal colourings of cellular and square grids is an immediate consequence of Theorem 1.

Theorem 2. *Given σ, and given a tiling of a cellular or square grid by tiles (for σ), a colouring with reuse distance σ is optimal iff all the tiles in the tiling are identical in their colour assignment.* □

Recall the definition of Tilings A and B from Section 3.2 (See Figures 4 and 6). From the proof of Theorem 1, we make the following observation:

Corollary 1. *Suppose $\sigma = 2k + 1$, and we have an optimal colouring of the cellular (square) grid. If for any point p in the grid, the vertex corresponding to position number k on an edge of the bounding box of p has the same colour as that assigned to p, then the tiling of the grid, by identically coloured tiles, corresponds to Tiling B; if the position number is $k + 1$, then the tiling of the grid corresponds to Tiling A.*

4 Optimal $L(\delta_1, \vec{1}_{\sigma-2})$ Colourings for $\mathcal{G}(\mathbf{A}_2)$ and $\mathcal{G}(\mathbf{Z}^2)$

In this section, we deal with optimal frequency assignment schemes for wireless networks modelled as cellular grids and square grids. We first present an $L(\delta_1, \vec{1}_{\sigma-2})$ colouring scheme of $\mathcal{G}(\mathbf{A}_2)$ for the case

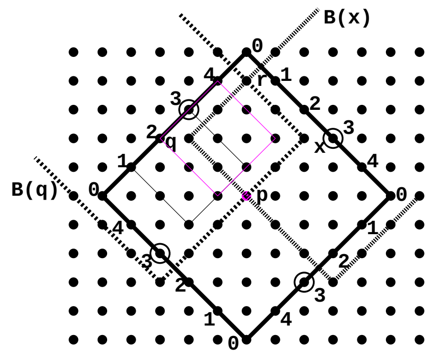

Figure 11: Square Grid *Bounding Box* for $\sigma = 5$ with position numbers

where *reuse distance* is odd i.e., $\sigma = 2k+1$, $k \in \{1, 2, \ldots\}$. This is followed by an $L(\delta_1, \vec{1}_{\sigma-2})$ colouring scheme of $\mathcal{G}(\mathbf{Z}^2)$ for all values of σ. The colouring schemes presented here correspond to Tiling A.

4.1 Cellular Grids

We present a colouring scheme where δ_1 varies as the square of σ, for $\sigma \geq 5$, σ odd. We note that the colouring of the entire cellular grid is achieved by colouring one tile and reproducing the same colouring in all the tiles present in the grid. Recall that the number of vertices $c(\sigma)$, in a tile corresponding to an odd reuse distance $\sigma = 2k+1$ is equal to $3k^2 + 3k + 1$ (see Lemma 1). From the above fact and from Theorem 2, we make the following observations: (refer to Figure 12)

Lemma 4. *1. Colouring $c(\sigma)$ points starting from the vertex of a tile along the direction j is equivalent to colouring all the diagonals of a tile in the following order:*

$$L_0, L_{k+1}, L_1, L_{k+2}, \ldots, L_{k-1}, L_{2k}, L_k.$$

2. *Along a line $i = m$, where m is a constant, any pair of points which are at a distance $c(\sigma)$ apart will have the same colour assigned to them.*

3. *Consider a point (p, q) on the line $i = p$. The point $(p+1, q-3k-1)$ on the line $i = p+1$ will have the same colour as (p, q).* □

From Lemma 4.1, we see that a colouring for $c(\sigma)$ points along a line $i = m$ for some arbitrary m describes the colouring for the entire grid.

The following colouring scheme is shown in Figure 12. To colour along the line $i = 0$, we proceed as follows: Starting with the point $(0, 0)$ which is assigned the colour 0, we assign consecutive colours to every third vertex, and wrap around after the $c(\sigma)^{th}$ vertex. This will colour all the $c(\sigma)$ points in three passes uniquely. This can be easily seen because $c(\sigma) = c'(k) = 3k^2 + 3k + 1 = 1(\bmod 3)$. Consecutive sets of $c(\sigma)$ vertices along this line follow the same colouring pattern.

Formally, this colouring scheme can be expressed as follows. Let $\chi(i, j)$ represent the colour assigned to the vertex (i, j) and $\chi'(j)$ represent the colour assigned to the vertex $(0, j)$, i.e. $\chi'(j) = \chi(0, j)$. We first give a formula for $\chi'(j)$ and then derive an expression for $\chi(i, j)$.

$$\chi'(j) = \begin{cases} \rho, & \bar{j} = 0 \pmod 3 \\ \rho + 2k^2 + 2k + 1, & \bar{j} = 1 \pmod 3 \\ \rho + k^2 + k + 1, & \bar{j} = 2 \pmod 3 \end{cases}$$

where $\bar{j} = j \bmod c(\sigma)$ and $\rho = \lfloor \frac{\bar{j}}{3} \rfloor$. Now, from Lemma 4.3, we can easily derive that

$$\chi(i,j) = \chi'\big(j + i(3k+1)\big)$$

Theorem 3. *For all $\sigma = 2k+1, k = \{1,2,\ldots\}$, the colouring scheme described above is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{A}_2)$, with $\delta_1 = k^2$. Moreover, this is a constant time colouring scheme.*

Proof. From Lemma 1, $c(\sigma)$ is a lower bound. We can easily see that each vertex in the tile is assigned a unique colour from the set $\{0, 1, \ldots, c(\sigma) - 1\}$. This implies that the optimality condition is satisfied.

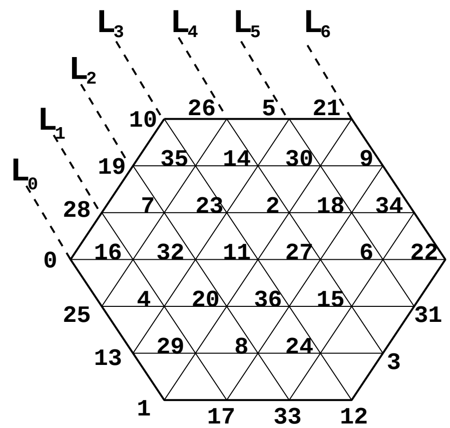

Figure 12: $L(\delta_1, \vec{1}_{\sigma-2})$ colouring for $\sigma = 7$

Again, the above scheme ensures that corresponding points in neighbouring tiles have the same colour and are exactly σ distance apart. Thus, the re-use constraint is satisfied.

To derive the value of δ_1, we proceed as follows. Consider a point (i,j) in the grid. Its six neighbours are $(i, j-1)$, $(i+1, j)$, $(i+1, j+1)$, $(i, j+1)$, $(i-1, j)$ and $(i-1, j-1)$.

The colour assigned to (i,j) according to the above scheme will be $\chi(i,j) = \chi'\big(j + i(3k+1)\big)$.

The table in Figure 13 shows the colours assigned to the neighbours of (i,j). (All the colour expressions are modulo $c(\sigma)$.)

From the table in Figure 13, we see that the least difference between the colours assigned to neighbouring points is k^2. Hence, $\delta_1 = k^2$.

From the formula for $\chi(i,j)$, it can be easily seen that given any arbitrary point (i,j) in the grid, the colour assigned to (i,j) can be computed in constant time. □

Lemma 5 notes the values of δ_2 and δ_3 for the above colouring.

Lemma 5. *For all $\sigma = 2k+1, k = \{1,2,\ldots\}$, the colouring scheme described above has the properties that $\delta_2 = k$ and $\delta_3 = 1$.*

$(i, j-1)$	$\big(\chi(i,j) + k^2 + k\big)$
$(i+1, j)$	$\big(\chi(i,j) + 2k^2 + 3k + 1\big)$
$(i+1, j+1)$	$\big(\chi(i,j) + k^2 + 2k + 1\big)$
$(i, j+1)$	$\big(\chi(i,j) + 2k^2 + 2k + 1\big)$
$(i-1, j)$	$\big(\chi(i,j) + k^2\big)$
$(i-1, j-1)$	$\big(\chi(i,j) + 2k^2 + k\big)$

Figure 13: Colours (mod $c(\sigma)$) assigned to neighbours of (i,j) by the scheme χ.

Proof. Similar to the proof of Theorem 3 above, using the table in Figure 13 twice proves the value of δ_2. The value of δ_3 is 1 by construction. □

4.2 Square Grids

We present colouring schemes where δ_1 varies as the square of σ, for $\sigma \geq 4$. There are two different schemes, one for the case where σ is odd and one for even σ. We note that the colouring of the entire square grid is achieved by colouring one tile and reproducing the same colouring in all the tiles present in the grid.

Odd σ

Recall that the number of vertices $c(\sigma)$, in a tile corresponding to an odd reuse distance $\sigma = 2k+1$ is equal to $2k^2+2k+1$ (see Lemma 2). From the above fact and from Theorem 2, we make the following observations: (refer to Figure 14)

Lemma 6. *1. Colouring $c(\sigma)$ points starting from the vertex of a tile along the direction j is equivalent to colouring all the diagonals of a tile in the following order:*

$$V_0, V_k, V_{2k}, V_{k-1}, \ldots, V_{k+2}, V_1, V_{k+1}.$$

2. Along a line $i = m$, where m is a constant, any pair of points which are at a distance $c(\sigma)$ apart will have the same colour assigned to them.

3. Consider a point (p, q) on the line $i = p$. The point $(p+1, q+2k+1)$ on the line $i = p+1$ will have the same colour as (p, q). □

From Lemma 6.1, we see that a colouring for $c(\sigma)$ points along a line $i = m$ for some arbitrary m describes the colouring for the entire grid.

The following colouring scheme is shown in Figure 14. To colour along the line $i = 0$, we proceed as follows: Starting with the point $(0,0)$ which is assigned the colour 0, we assign consecutive colours to every second vertex, and wrap around after the $c(\sigma)^{th}$ vertex. This will colour all the $c(\sigma)$ points in two passes uniquely. This can be easily seen because $c(\sigma) = c'(k) = 2k^2 + 2k + 1$ is odd and hence, points coloured in the first pass will not be repeated again. Consecutive sets of $c(\sigma)$ vertices along this line follow the same colouring pattern.

Mathematically, this colouring scheme can be expressed as follows. Let $\Omega(i, j)$ represent the colour assigned to the vertex (i, j) and $\Omega'(j)$ represent the colour assigned to the vertex $(0, j)$, i.e. $\Omega'(j) = \Omega(0, j)$. We first give a formula for $\Omega'(j)$ and then derive an expression for $\Omega(i, j)$.

$$\Omega'(j) = \begin{cases} \rho, & \bar{j} \text{ even} \\ \rho + k^2 + k + 1, & \bar{j} \text{ odd} \end{cases}$$

where $\bar{j} = j \bmod c(\sigma)$ and $\rho = \lfloor \frac{\bar{j}}{2} \rfloor$. Now, from Lemma 6.3, we can easily derive that

$$\Omega(i, j) = \Omega'\big(j - i(2k+1)\big)$$

Theorem 4. *For all $\sigma = 2k+1, k = \{1, 2, \ldots\}$, the colouring scheme described above is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{Z}^2)$, with $\delta_1 = k^2$. Moreover, this is a constant time colouring scheme.*

Proof. From Lemma 2, $c(\sigma)$ is a lower bound. We can easily see that each vertex in the tile is assigned a unique colour from the set $\{0, 1, \ldots, c(\sigma) - 1\}$. This implies that the optimality condition is satisfied.

Again, the above scheme ensures that corresponding points in neighbouring tiles have the same colour and are exactly σ distance apart. Thus, the re-use constraint is satisfied.

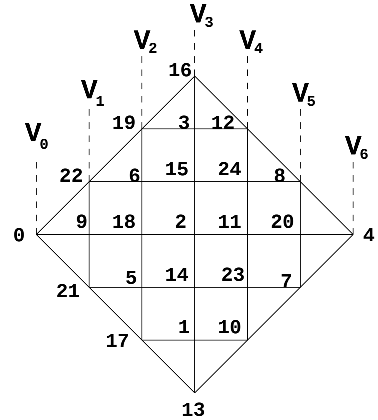

Figure 14: $L(\delta_1, \vec{1}_{\sigma-2})$ colouring for $\sigma = 7$

$(i-1,j)$	$\big(\Omega(i,j) + k^2 + 2k + 1\big)$
$(i,j+1)$	$\big(\Omega(i,j) + k^2 + k + 1\big)$
$(i+1,j)$	$\big(\Omega(i,j) + k^2\big)$
$(i,j-1)$	$\big(\Omega(i,j) - k^2 - k - 1\big)$

Figure 15: Colours (mod $c(\sigma)$) assigned to the neighbours of (i,j) by the scheme Ω when σ is odd.

To derive the value of δ_1, we proceed as follows. Consider a point (i,j) in the grid. Its four neighbours are $(i-1,j)$, $(i,j+1)$, $(i+1,j)$, and $(i,j-1)$.

The colour assigned to (i,j) according to the above scheme will be $\Omega(i,j) = \Omega'\big(j - i(2k+1)\big)$.

The table in Figure 15 shows the colours (modulo $c(\sigma)$) assigned to the neighbours of (i,j).

From the table in Figure 15, we see that the least difference between the colours assigned to neighbouring points is k^2. Hence, $\delta_1 = k^2$.

From the formula for $\Omega(i,j)$, it can be easily seen that given any arbitrary point (i,j) in the grid, the colour assigned to (i,j) can be computed in constant time.

□

Even σ

We now present a colouring scheme for even $\sigma, \sigma \geq 4$, i.e. $\sigma = 2k$, $k \in \{2, 3, \ldots\}$. We first note that the total number of points in a tile in terms of k will be equal to $2k^2$. Since colouring of the entire grid is achieved by colouring one tile and reproducing the same colouring in all tiles of the grid, description of the colouring for a single tile is sufficient.

The colouring scheme is shown in Figure 16. Alternate diagonals are coloured consecutively starting with D_0, i.e. the following diagonals D_0, D_2, $\ldots$, $D_{\sigma-2}$, D_1, D_3, $\ldots$, $D_{\sigma-1}$ are coloured in order. Starting with the origin of the tile which is assigned colour 0, points are coloured consecutively within each diagonal.

Let $\Omega(i,j)$ be the colour assigned to the point (i,j) in the grid. It can be mathematically expressed as follows:

$$\Omega(i,j) = \lfloor\frac{(i-j) \bmod 2k}{2}\rfloor k + \lfloor\frac{(i+j) \bmod 2k}{2}\rfloor + ((i+j) \bmod 2)k^2$$

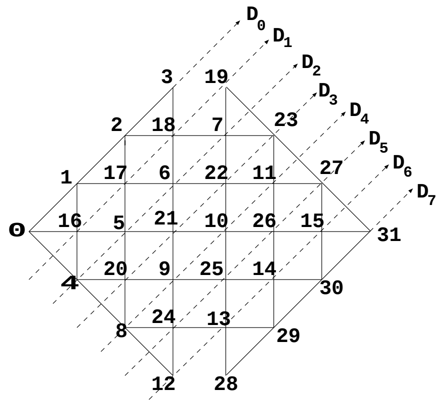

Figure 16: $L(\delta_1, \vec{1}_{\sigma-2})$ colouring for $\sigma = 8$

$(i-1, j)$	$\Omega(i,j) - k^2$
$(i, j+1)$	$\Omega(i,j) - k^2 - k + 1$, if $(i+j) \bmod 2k = 2k-1$ $\Omega(i,j) - k^2 + 1$, otherwise
$(i+1, j)$	$\Omega(i,j) - k^2 + 1$, if $(i+j) \bmod 2k = 2k-1$ $\Omega(i,j) - k^2 + k + 1$, otherwise
$(i, j-1)$	$\Omega(i,j) - 2k^2 + k$, if $(i-j) \bmod 2k = 2k-1$ $\Omega(i,j) - k^2 + k$, otherwise

Figure 17: Colours (mod $c(\sigma)$) assigned to the neighbours of (i,j) by the scheme Ω when σ is even.

Theorem 5. *For all $\sigma = 2k, k = \{2, 3, \ldots\}$, the colouring scheme described above is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{Z}^2)$, with $\delta_1 = k^2 - k - 1$. Moreover, this is a constant time colouring scheme.*

Proof. From Lemma 2, $c(\sigma)$ is a lower bound. We can easily see that each vertex in the tile is assigned a unique colour from the set $\{0, 1, \ldots, c(\sigma) - 1\}$. This implies that the optimality condition is satisfied.

Again, from the formula, we see that corresponding points in neighbouring tiles have the same colour and are exactly σ distance apart. Thus, the re-use constraint is satisfied.

To derive the value of δ_1, we proceed as follows. Consider the neighbours of an arbitrary point (i,j) in the grid. They are $(i-1, j)$, $(i, j+1)$, $(i+1, j)$ and $(i, j-1)$. We will find the differences between the colours assigned to (i,j) and each of its neighbours. The least difference will be equal to δ_1.

There are two cases to consider: 1) $(i+j)$ is even, and 2) $(i+j)$ is odd. Note that $(i+j)$ value for alternate points in both X and Y directions will be of the same parity. If we consider a point for which $(i+j)$ is odd, $(i+j)$ for all its neighbours will be even and vice versa. It follows that we need to consider only one case, as considering the other case will yield the same expressions for the differences.

Consider a point (i,j) and suppose $(i+j)$ is odd. Let the colour assigned to (i,j) be $\Omega(i,j)$. The table in Figure 17 shows the colours assigned to the neighbours of (i,j).

Clearly, from the table in Figure 17, the least difference between the colours assigned to neighbouring points is $k^2 - k - 1$. Hence, $\delta_1 = k^2 - k - 1$.

From the formula for $\Omega(i, j)$, it can be easily seen that given any arbitrary point (i, j) in the grid, the colour assigned to (i, j) can be computed in constant time.

□

5 Upper Bound on δ_1

The previous subsections presented colouring schemes for odd reuse distances where δ_1, the channel separation constraint has a value of k^2, where $\sigma = 2k + 1$. Lemma 7 provides an upper bound on δ_1.

Lemma 7. *For all $\sigma = 2k+1, k = \{1, 2, \ldots\}$, for any optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{A}_2)$ ($\mathcal{G}(\mathbf{Z}^2)$, respectively) $\delta_1 < k^2 + k$.*

Proof. Suppose C is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{A}_2)$. Since there are $3k^2 + 3k + 1$ vertices in a tile of $\mathcal{G}(\mathbf{A}_2)$, C assigns each number in $\{0, \ldots, 3k^2 + 3k\}$ to some vertex in $\mathcal{G}(\mathbf{A}_2)$. Let $\delta_1 = \delta$ be the separation between the colours assigned by C to any two adjacent vertices.

Consider the vertex, v, assigned the colour $\delta - 1$. Each of its six neighbours must be assigned a colour that is at least $2\delta - 1$. Since (1) the neighbours of v form a cycle of length 6, and (2) each adjacent pair of vertices in the cycle must be assigned colours differing by at least δ, it follows that at least three of these vertices must each be assigned a colour that is at least $3\delta - 1$. Thus, at least one neighbour of v must be assigned a colour that is at least $3\delta + 1$. Then,

$$3\delta + 1 \leq 3k^2 + 3k \Longrightarrow \delta < k^2 + k.$$

A similar argument can be used to show that for an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring of $\mathcal{G}(\mathbf{Z}^2)$, $\delta < k^2 + k$. □

If $\sigma = 2k$ is even, the size of the tile in $\mathcal{G}(\mathbf{A}_2)$ and in $\mathcal{G}(\mathbf{Z}^2)$ is $3k^2$ and $2k^2$, respectively. Then, an argument similar to the one in the proof of Lemma 7 can be used to show that

Lemma 8. *For all $\sigma = 2k, k = \{1, 2, \ldots\}$, for any optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{A}_2)$ ($\mathcal{G}(\mathbf{Z}^2)$, respectively) $\delta_1 < k^2$.*

Based on experimental verification by means of an exhaustive search for all values of $k \leq 4$, we conjecture that

Conjecture 1. *For all σ, for any optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{A}_2)$ ($\mathcal{G}(\mathbf{Z}^2)$, respectively) $\delta_1 \leq (\lfloor \sigma/2 \rfloor)^2$.*

The conjecture implies a tighter upper bound, for odd σ, than the one presented in Lemma 7 above. Note that, for odd σ, our assignments presented in Sections 4.1 and 4.2 do realise this value for δ_1.

6 Conclusions and Open Problems

We characterised optimal channel assignment schemes for cellular and square grids, and hence showed that any such scheme must be uniform across the entire grid. More specifically, in an optimal colouring, the colouring of a tile (for a given σ) will be identically repeated in all the tiles throughout the grid. We also presented optimal $L(\delta_1, \vec{1}_{\sigma-2})$ colouring schemes, with a high value for δ_1, for square grids for all $\sigma \geq 4$ and for cellular grids for the case where *reuse distance* is odd i.e., $\sigma = 2k + 1$, $k \in \{1, 2, \ldots\}$. The previous best known results have been restricted to $\delta_1 \lessapprox \frac{3k^2}{8}$ [6], in case of cellular grids and $\delta_1 \leq \lfloor \frac{\sigma - 1}{2} \rfloor$ [3] in case of square grids. We prove an upper bound for δ_1 for optimal colourings of cellular and square grids. In the case of σ being odd, we conjecture that our value of δ_1 is a tight upper bound on δ_1 for optimal colouring schemes for these grids.

Several interesting open questions arise from the work presented here. We list a few of them here: (1) Find optimal colouring schemes for cellular grids with high δ_1 values for the case when σ is even. (2) Find and prove the existence of tight upper bounds for $\delta_1, \delta_2, \ldots$ for a general σ.

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