Channel Assignment in Honeycomb Networks *

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Abstract. The honeycomb grid is a network topology based on the hexagonal plane tessellation, which is convenient to model the regular placement on the plane of the base stations of wireless networks. For an efficient use of the radio spectrum in such networks, channels have to be assigned to the base stations so as to avoid interferences. Such a problem can be modeled as a suitable coloring problem. Precisely, given an integer t and a honeycomb grid $G = (V, E)$, an $L(1^t)$-coloring of G is a function f from the vertex set V to a set of nonnegative integers such that $|f(u) - f(v)| \geq 1$, if the distance between the vertices u and v is at most t. This paper presents efficient algorithms for finding optimal $L(1^t)$-colorings of honeycomb grids.

1 Introduction

In the 4th generation of wireless access systems, due to the decreasing cost of infrastructures and to the need of wider bandwidth, a large number of small cells, each with significant power, is expected to cover a huge communication region [16]. Such a covering can be achieved by placing the base stations according to a regular plane tessellation. It is well-known that only three different regular tessellations of the plane exist, depending on the kind of regular polygons used. Specifically, the honeycomb, square and hexagonal tesselations cover the plane, respectively, by regular hexagons, squares, and triangles. Such tessellations can be used to place at the polygon vertices the base stations of the wireless communication networks, leading to three well-known topologies: *honeycomb*, *square* and *hexagonal* grids, depicted in Fig. 1 for 16 vertices.

So far, the most studied topology for wireless communication networks has been the hexagonal grid [3, 9, 12]. However, the performance of a topology can be evaluated with respect to several parameters, such as *degree* and *diameter*. As proved in [14], defined the network *cost* as the product of the degree and diameter, the honeycomb grid beats both the hexagonal and square grids, as summarized in Table 1 for grids with n vertices (in such a table, coefficients are rounded and additive constants are neglected). Therefore, the honeycomb grid appears more convenient than the hexagonal and square grids to model the placement of the base stations.

In a wireless network, the main difficulty against an efficient use of the radio spectrum is given by *interferences*, which result in damaged communications. Interferences can be eliminated by means of suitable *channel assignment* techniques, which partition the given radio spectrum into a set of disjoint channels.

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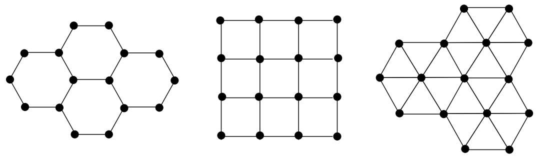

Fig. 1. The possible grids of 16 vertices: (a) honeycomb, (b) square, and (c) hexagonal.

The same channel can be reused by two stations at the same time provided that no interference arises. To avoid interference, a *separation vector* $(\delta_1, \delta_2, \ldots, \delta_t)$ of non increasing positive integers is introduced in such a way that channels assigned to interfering stations at distance i be at least δ_i apart, with $1 \leq i \leq t$, while the same channel can be reused only at stations whose distance is larger than t [8, 9]. Since only a continuous interval of the radio spectrum can be acquired, the objective is to minimize its width (or span), namely the difference between the highest and lowest channels assigned. In case of separation vectors containing repeated integer values, a more compact notation will be convenient and so, as an example, $(\delta_1, 1^q)$ is a shorthand for $(\delta_1, \underbrace{1, 1, \ldots, 1}_{q})$.

Formally, given $(\delta_1, \delta_2, \ldots, \delta_t)$ and an undirected graph $G = (V, E)$, an $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of G is a function f from the vertex set V to the set of nonnegative integers $\{0, \ldots, \lambda\}$ such that $|f(u) - f(v)| \geq \delta_i$, if $d(u, v) = i$, $1 \leq i \leq t$, where $d(u, v)$ is the distance between vertices u and v. An *optimal* $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring for G is one minimizing λ over all such colorings. Thus, the channel assignment problem consists of finding an optimal $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring for G.

The $L(1^t)$-coloring problem has been widely studied in the past [1, 6, 10, 12]. In particular, its intractability has been proved by McCormick [10], while optimal $L(1^t)$-colorings have been proposed in [1, 2] for rings, trees, and square grids. Moreover, optimal $L(\delta_1, 1^{t-1})$-colorings have been proposed in [3, 13] for rings, square grids and hexagonal grids. Optimal $L(\delta_1, \delta_2)$-colorings on square grids and hexagonal grids have been given by Van Den Heuvel et al. [15], who provided also an optimal $L(2, 1^2)$-coloring for square grids. The $L(2, 1^2)$-coloring

network	**degree**	**diameter**	**cost**
honeycomb grid	3	$1.63\sqrt{n}$	$4.9\sqrt{n}$
square grid	4	$2\sqrt{n}$	$8\sqrt{n}$
hexagonal grid	6	$1.16\sqrt{n}$	$6.93\sqrt{n}$

Table 1. [14] Comparison of networks, each with n vertices (data are approximated).

problem has been also optimally solved for hexagonal grids and rings in [3]. Finally, the $L(2,1)$-coloring problem has been studied also in [4, 5, 7, 11].

This paper provides, for the first time, optimal $L(1^t)$-colorings on honeycomb grids. Such colorings use less colors than those needed by the hexagonal and square grids. Therefore, honeycomb grids beat the hexagonal and square grids in terms of both the network cost and channel requirement. The proposed algorithms allow any vertex to self-assign its proper channel in constant time, provided that it knows its relative position within the network. If this is not the case, such relative positions can be computed for all the vertices using simple distributed algorithms requiring optimal time and optimal number of messages, as explained in [3].

2 Preliminaries

The $L(1)$-coloring problem on a graph G is exactly the classical vertex coloring problem on G, where the minimum number of colors needed is $\lambda + 1 = \chi(G)$, the *chromatic number* of G. In the case of $L(1^t)$-colorings, the term *t-chromatic number* of G, denoted by $\chi_t(G)$, will be used. A lower bound for $\chi_t(G)$ is the size $\omega(A_{G,t})$ of the *maximum clique* of the *augmented* graph $A_{G,t}$, which has the same vertex set as G and the edge $[r,s]$ iff $d(r,s) \leq t$ in G.

A *t-independent set* is a subset S_t of vertices of G whose pairwise distance is at least $t+1$. If the size of S_t is the largest possible, then S_t is a *maximum t-independent set*, and is denoted by S_t^*. Assigning different colors to different t-independent sets one obtains a feasible $L(1^t)$-coloring. Conversely, given a feasible $L(1^t)$-coloring, all the vertices with the same color form a t-independent set. Any feasible $L(1^t)$-coloring uses at least as many colors as the minimum number $\mu_t(G)$ of maximum t-independent sets that cover all the vertices, that is $\chi_t(G) \geq \mu_t(G)$.

Let G_1 and G_2 be any two graphs, and let $V(G)$ denote the vertex set of a graph G. A *t-homomorphism* from G_1 to G_2 is a total function $\phi : V(G_1) \mapsto V(G_2)$ such that: (i) $\phi(u) = \phi(v)$ only if $u = v$ or $d(u,v) > t$, and (ii) $d(\phi(u), \phi(v)) \leq d(u,v)$ for all nodes u, v of G_1. Now, if g is an $L(\delta_1, \ldots, \delta_t)$-coloring of G_2, and ϕ is a t-homomorphism from G_1 to G_2, then the composition $g \circ \phi$ is an $L(\delta_1, \ldots, \delta_t)$-coloring of G_1.

In this paper, *brick* representations of honeycomb grids are adopted where each hexagon is represented by a rectangle spanning 3 rows and 2 columns. In this way, a honeycomb grid H of size $n = rc$ is represented by r rows and c columns, indexed respectively from 0 to $r-1$ (from top to bottom) and from 0 to $c-1$ (from left to right), with $r \geq 3$ and $c \geq 2$. A generic vertex u of H is denoted by $u = (i,j)$, where i is its row index and j is its column index. Note that each vertex (i,j), which does not belong to the grid borders, has degree 3 and is adjacent to the following 3 vertices: $(i-1,j)$, $(i+1,j)$, and $(i,j+1)$ if $i+j$ is even, or $(i,j-1)$ if $i+j$ is odd.

3 Optimal $L(1^t)$-coloring

In this section, optimal $L(1^t)$-colorings of sufficiently large honeycomb grids will be presented, which depend on the parity of t. In particular, when t is odd the lower bound on $\chi_t(H)$ is given by $\omega(A_{H,t})$ and this bound is achievable. When t is even, such a lower bound is not achievable, and a stronger lower bound is needed which depends on $\mu_t(H)$. In both cases, the optimal colorings are based on a grid tessellation.

3.1 $L(1^t)$-coloring with t odd

Lemma 1. *Let $t = 8p + q$, with $p \geq 0$ and $q = 1, 3, 5, 7$. There is an $L(1^t)$-coloring of a honeycomb grid H of size $r \times c$, with $r \geq t + 1$ and $c \geq \lceil \frac{t-3}{4} \rceil + \lfloor \frac{t+1}{4} \rfloor + 1$, only if*

$$\chi_t(H) \geq \omega(A_{H,t}) = \begin{cases} 24p^2 + 12p + 2 & \textit{if } q = 1 \\ 24p^2 + 24p + 6 & \textit{if } q = 3 \\ 24p^2 + 36p + 14 & \textit{if } q = 5 \\ 24p^2 + 48p + 24 & \textit{if } q = 7 \end{cases}$$

Proof The maximum clique of $A_{H,t}$ is a *diamond* with $\lceil \frac{t-3}{4} \rceil + \lfloor \frac{t+1}{4} \rfloor + 1$ columns. The leftmost column has $(t+1) - 2\lceil \frac{t-3}{4} \rceil$ vertices, and each subsequent column has two extra vertices up to the central column which counts $t+1$ vertices. Each of the remaining $\lfloor \frac{t+1}{4} \rfloor$ columns, on the right of the central one, decreases its size by two. In particular, the rightmost column has $(t+1) - 2\lfloor \frac{t+1}{4} \rfloor$ vertices. Depending on the value of q, the number of left and right columns is, respectively:

$$\left\lceil \frac{t-3}{4} \right\rceil = \begin{cases} 2p & \text{if } q = 1, 3 \\ 2p + 1 & \text{if } q = 5, 7 \end{cases} \qquad \left\lfloor \frac{t+1}{4} \right\rfloor = \begin{cases} 2p & \text{if } q = 1 \\ 2p + 1 & \text{if } q = 3, 5 \\ 2p + 2 & \text{if } q = 7 \end{cases}$$

Note that the shape of the maximum clique varies with q. For instance, Fig. 2 shows the maximum cliques when $q = 1, 3, 5$ and 7 and $t = 17, 19, 21$ and 23, respectively. Then,

$$\omega(A_{H,t}) = (t+1) + \sum_{i=1}^{\lfloor \frac{t+1}{4} \rfloor} (t + 1 - 2i) + \sum_{i=1}^{\lceil \frac{t-3}{4} \rceil} (t + 1 - 2i)$$

Solving the above formula with $t = 8p + q$, the proof follows. ∇

By the above lemma, all the vertices of each diamond must get a different color. An optimal $L(1^t)$-coloring, with t odd, can be easily achieved tessellating the honeycomb grid by means of diamonds, all colored in the same way. Observing Fig. 2, one notes that diamonds have the same number of left and right columns, i.e. they are symmetric, for $q = 1, 5$; while they have one more right column, i.e., they are asymmetric, for $q = 3, 7$. Therefore, there are two possible tessellations depending on the symmetry of the diamonds, which are illustrated in Figure 3 (where $t = 13$ and $t = 15$ are assumed).

In the following, it is shown how a color can be assigned in constant time to any vertex u of the grid. The coloring depends on the symmetry of the diamond.

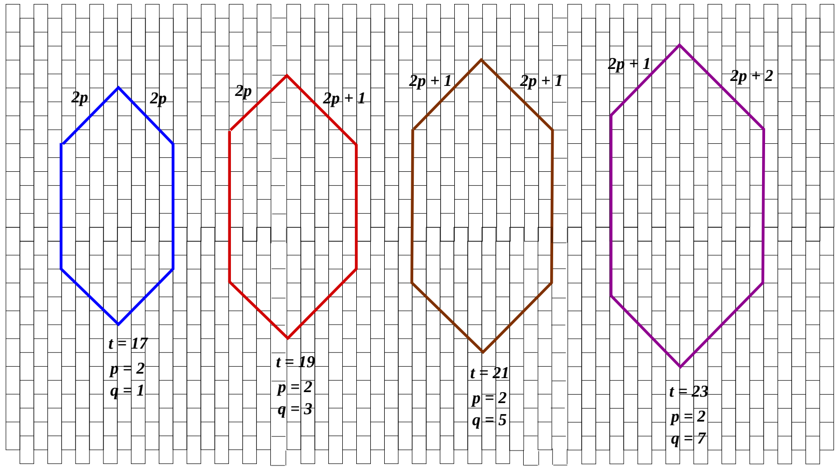

Fig. 2. The maximum cliques (diamonds) for $t = 17, 19, 21$ and 23.

Coloring with symmetric diamonds in $O(1)$ time Consider the case with $t = 8p + 1$ (the other case with $t = 8p + 5$ can be dealt with similarly). The diamond has as many right columns as its left columns, namely $2p$ (see the leftmost diamond in Fig. 2).

Observe the honeycomb tessellation by the symmetric diamonds, and restrict the attention to the rectangle $\mathcal{R}$, consisting of the leftmost $4p + 1$ columns and the uppermost $\omega(A_{H,8p+1}) = 24p^2 + 12p + 1$ rows of the grid, as depicted in Fig. 4 (left) for $t = 9$, namely $p = 1$. Clearly, the top left corner of $\mathcal{R}$ has coordinates $(0, 0)$. Sequentially scanning top-down the vertices in column 0 of $\mathcal{R}$, $4p + 1$ different diamonds are encountered. Moreover, for each traversed diamond, a different column is encountered and overall all the $4p + 1$ diamond columns, and hence all the diamond vertices, are met. By the above property, assigning a different color to each vertex in column 0 of $\mathcal{R}$ allows each diamond within the tessellation to be colored the same using the minimum number of colors.

To achieve such a goal, let the diamond columns be numbered from left to right, starting from 0 and ending at $4p$. The diamond columns met along column 0 of $\mathcal{R}$ follow the order shown in Table 2. Formally, denoted by $x(j)$ the order in which the diamond column j is encountered, $x(j) = j(4p - 1) \bmod (4p + 1)$. Conversely, given the order x in which a column is encountered, the column index $col(x) = 2px \bmod (4p + 1)$.

order	0	1	2	3	4	...	$4p-1$	$4p$
column	0	$2p$	$4p$	$2p-1$	$4p-1$	...	1	$2p+1$

Table 2. The order in which diamond columns are encountered.

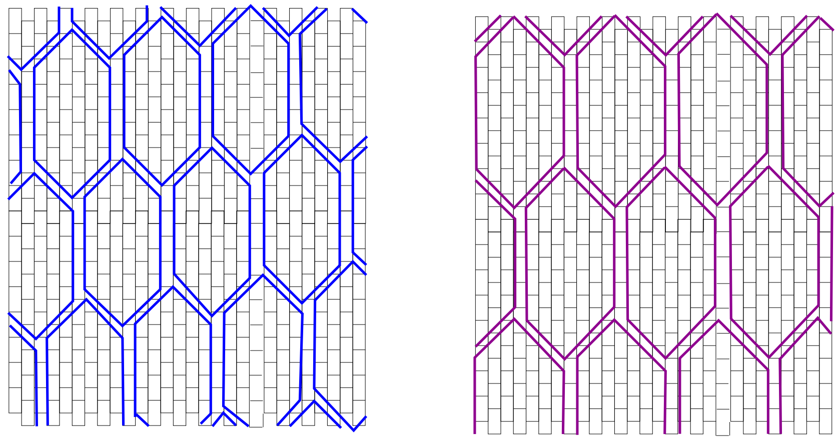

Fig. 3. The honeycomb tessellation: on the left, by the symmetric diamonds ($t = 13$); on the right, by the asymmetric diamonds ($t = 15$).

Moreover, the size, i.e. the number of vertices, of the diamond column j is:

$$size(j) = \begin{cases} 4p + 2 + 2j & \text{if } 0 \leq j \leq 2p \\ 12p + 2 - 2j & \text{if } 2p \leq j \leq 4p \end{cases}$$

Finally, the number of vertices of a diamond that have been encountered before the topmost vertex of column j is:

$$pred(j) = \sum_{k=0}^{x(j)-1} size(col(k)).$$

As an example, $x(j), size(j)$ and $pred(j)$ are also shown in Fig. 4 for $t = 9$ (i.e. $p = 1$).

Now, in order to assign different colors to all the vertices in column 0 of $\mathcal{R}$, let the color of vertex $(i, 0)$ be simply $g(i, 0) = i$. The coloring of the entire rectangle $\mathcal{R}$ is obtained assigning to the remaining columns a suitable cyclic shift of the coloring of column 0. Such a cyclic shift is chosen so that all the diamond columns with the same number are colored the same in all diamonds. To do this, let the shift for column j be denoted by $\Delta(j)$. Given the above coloring for column 0 of $\mathcal{R}$, it is easy to see that $\Delta(j)$, where $0 \leq j \leq 4p$, must be:

$$\Delta(j) = \begin{cases} (pred(j) + 2p) - (2p - j) & \text{if } 0 \leq j \leq 2p \\ (pred(j) + 2p) - (j - 2p) & \text{if } 2p \leq j \leq 4p \end{cases}$$

In conclusion, given any vertex $(i, j) \in \mathcal{R}$, its color is defined as

$$g(i, j) = (\Delta(j) + i) \bmod (24p^2 + 12p + 1).$$

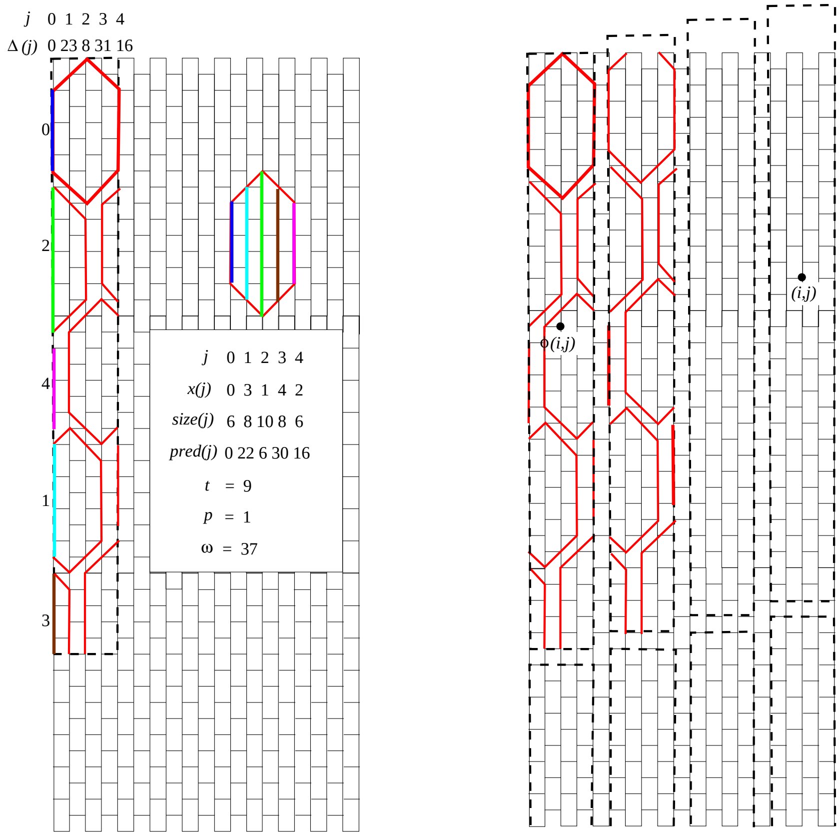

Fig. 4. The diamond columns encountered by scanning column 0 of $\mathcal{R}$ (left). The coloring of H by copies of $\mathcal{R}$ (right).

The coloring of the entire grid H is obtained by defining a t-homomorphism $\phi : V(H) \mapsto V(\mathcal{R})$, which can be viewed as a covering of H with colored copies of $\mathcal{R}$. Such copies are shifted up by one row, to reproduce, within each rectangle, the same diamond pattern as in $\mathcal{R}$, as shown in Fig. 4 (right). Hence, for any vertex $(i, j) \in H$, its color $f(i, j)$ is given by $g(\phi(i, j))$, where

$$\phi(i, j) = \left(\left(i + \left\lfloor \frac{j}{4p+1} \right\rfloor \right) \bmod (24p^2 + 12p + 1), j \bmod (4p+1) \right).$$

Observe that $\left\lfloor \frac{j}{4p+1} \right\rfloor$ counts how many rows the rectangle to which (i, j) belongs is shifted up with respect to the leftmost rectangle containing row i. Clearly, if $(i, j) \in \mathcal{R}$ then $\phi(i, j) = (i, j)$ and thus $f(i, j) = g(i, j)$. The correctness easily follows since all diamonds are colored the same, while the t-homomorphism provides a constant time coloring of each vertex which depends only on the vertex indices. Observe also that $\mathcal{R}$ contains $O(p^3)$ vertices, and that the computation of each $\Delta(j)$ requires $O(p)$ time. Since t (and hence p) is a costant, the coloring of vertex (i, j) takes $O(1)$ time.

Coloring with asymmetric diamonds in $O(1)$ time Consider now the case with $t = 8p + q$, where $q = 3, 7$. The diamond has one more right column than its left columns (see Fig. 2). Due to the fact that the diamonds are horizontally aligned in the tessellation (see Fig. 3), the coloring is much simpler than in the symmetric case. Let $left$ and $right$ denote the number of left and right columns, respectively. As one can check in the proof of Lemma 1:

$$left = \begin{cases} 2p & \text{if } q = 3 \\ 2p+1 & \text{if } q = 7 \end{cases} \qquad right = \begin{cases} 2p+1 & \text{if } q = 3 \\ 2p+2 & \text{if } q = 7 \end{cases}$$

Observe the honeycomb tessellation by the asymmetric diamonds, and restrict the attention to the rectangle $\mathcal{R}$, consisting of the leftmost $left+right+1$ columns and the uppermost $t - left - 1$ rows of the grid. As before, the top left corner of $\mathcal{R}$ is vertex $(0, 0)$. The number of vertices in $\mathcal{R}$ is exactly $\omega(A_{H,t})$, where

$$\omega(A_{H,t}) = \begin{cases} 24p^2 + 24p + 5 & \text{if } q = 3 \\ 24p^2 + 48p + 23 & \text{if } q = 7 \end{cases}$$

Any coloring of the grid H obtained by covering H by colored copies of $\mathcal{R}$ leads to a feasible and optimal coloring (see Fig. 5). Let $\mathcal{R}$ be colored in row-major order. In details, given any vertex $(i, j) \in \mathcal{R}$, let its color be

$$g(i, j) = (i(left + right + 1) + j) \bmod \omega(A_{H,t}).$$

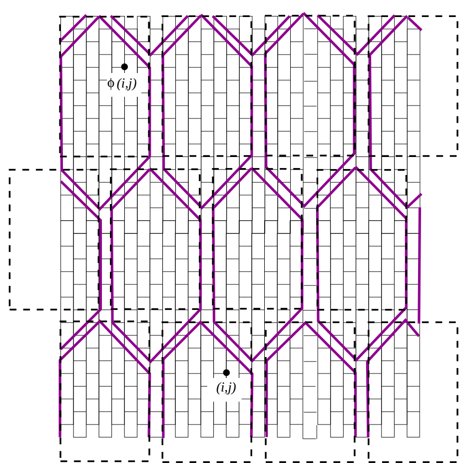

Fig. 5. The coloring of H by copies of $\mathcal{R}$ in the asymmetric case.

The coloring of the entire grid H is obtained by defining a t-homomorphism $\phi : V(H) \mapsto V(\mathcal{R})$. For any vertex $(i,j) \in H$, $f(i,j) = g(\phi(i,j))$ where

$$\phi(i,j) = \left(i \bmod (t-left), \left(j - right\left\lfloor \frac{i}{t-left} \right\rfloor\right) \bmod (left+right+1)\right)$$

3.2 $L(1^t)$-coloring with t even

Lemma 2. *Let $t = 8p + q$, with $p \geq 0$ and $q = 0, 2, 4, 6$. There is an $L(1^t)$-coloring of a honeycomb grid H of size $r \times c$, with $r \geq t+1$ and $c \geq \lfloor \frac{t}{4} \rfloor + \lceil \frac{t}{4} \rceil + 1$, only if*

$$\chi_t(H) \geq \omega(A_{H,t}) = \begin{cases} 24p^2 + 6p + 1 & \text{if } q = 0 \\ 24p^2 + 18p + 4 & \text{if } q = 2 \\ 24p^2 + 30p + 10 & \text{if } q = 4 \\ 24p^2 + 42p + 19 & \text{if } q = 6 \end{cases}$$

Proof As in Lemma 1, the maximum clique of $A_{H,t}$ is again a diamond, which can be symmetric or asymmetric. However, there are some *holes* (i.e., vertices not included in the clique) on a single border column of the diamond. The holes are located according to the *center* of the diamond. The center is the middle vertex (i,j) of the central column, which can be termed either *left center* or *right center* depending on whether it is horizontally connected either to vertex $(i, j-1)$ or $(i, j+1)$, respectively. In the symmetric case, the holes are located in the furthest column on the opposite side with respect to the horizontal connection of the center. Instead, in the asymmetric case, the holes are located on the same side as the center connection.

To compute the clique size $\omega(A_{H,t})$, a reasoning similar to that in the proof of Lemma 1 is followed. As an example, Fig. 6 shows the maximum cliques when $q = 0, 2, 4$ and 6 and $t = 16, 18, 20$ and 22, respectively (in such a figure, the clique centers are depicted by black dots). ∇

A stronger lower bound based on t-independent sets In contrast to the case t odd, when t is even the lower bound on the number of colors given by $\omega(A_{H,t})$ is no more reachable by an $L(1^t)$-coloring. Indeed it is possible to derive a stronger lower bound by considering how a maximum t-independent set S_t^* can be built.

Lemma 3. *When t is even, the minimum distances among three closest vertices belonging to S_t^* are $t+1, t+1$ and $t+2$.*

Proof Let a vertex $u = (i,j)$ of H be a *left vertex*, if it is horizontally connected to vertex $s = (i, j-1)$, or a *right vertex* if it is connected to $d = (i, j+1)$. By contradiction, assume there are 3 vertices u, v and w such that $d(u,v) = d(v,w) = d(w,u) = t+1$. W.l.o.g., let u be a left vertex. Since $t+1$ is odd, then both v and w must be right vertices. This implies that $d(v,w)$ must be even and greater than t. But this is a contradiction, and $d(v,w) = t+2$. ∇

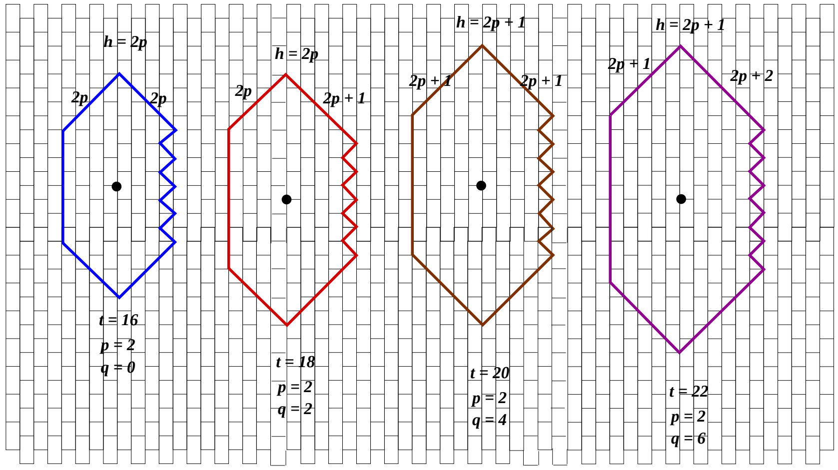

Fig. 6. The maximum cliques (diamonds) for $t = 16, 18, 20$ and 22. The number of holes is denoted by h.

Given a vertex v, let $B_t(v)$ be the set of vertices at distance exactly $t+1$ from v. One can show that $|B_t(v)| = 3t+3$ for any t even. To build a maximum t-independent set that contains v, let select as many vertices as possible among those in $B_t(v)$. By Lemma 3, since those vertices are all at distance $t+1$ from v, they must be at distance at least $t+2$ among them.

Lemma 4. *When t is even, there is no way to select 6 vertices $u_0, u_1, \ldots, u_5$ of $B_t(v)$ such that $d(u_i, u_{(i+1) \bmod 6}) = t+2$.*

Proof Since any two consecutive vertices of $B_t(v)$ are at distance 2, no more than one out of $\frac{t+2}{2}$ consecutive vertices of $B_t(v)$ can be selected. Therefore at most $\left\lfloor \frac{3t+3}{\frac{t+2}{2}} \right\rfloor = 5$ vertices can be selected. ∇

Lemma 5. *When t is even, there is no way to select 6 vertices $u_0, u_1, \ldots, u_5$ such that: (1) u_i belongs to $B_t(v)$ for $i = 1, \ldots, 5$; (2) $d(u_i, u_{i+1}) = t+2$, for $i = 1, \ldots, 4$; and (3) $d(u_0, u_1) = d(u_0, u_5) = t+1$.*

Proof After selecting $u_1, \ldots, u_5$ on $B_t(v)$ such that $d(u_i, u_{i+1}) = t+2$ for $i = 1, \ldots, 4$, there are $t-2$ vertices of $B_t(v)$ left out between u_5 and u_1. Then, there are t vertices at distance $t+2$ from v between u_5 and u_1. Moreover, the shortest path from u_5 to u_1, which does not include any other vertex of $B_t(v)$, has length $2t$. Therefore, there is no way to choose on such a path any vertex u_0 at distance $t+1$ from both u_0 and u_5. ∇

By the previous lemmas, to build a maximum t-independent set including a given vertex v, one should choose the six vertices closest to v such that at most four of them are at distance $t+1$ from v, and at least two of them are at distance $t+2$. Moreover, in a maximum t-independent set such a property should hold for any of its vertices, and in particular for the 6 vertices closest to v.

Lemma 6. *Let $t = 8p + q$, with $p \geq 0$ and $q = 0, 2, 4, 6$. The minimum number of maximum t-independent sets that cover a sufficiently large honeycomb grid H is:*

$$\mu_t(H) \geq \begin{cases} 24p^2 + 8p + 1 & \text{if } q = 0 \\ 24p^2 + 20p + 4 & \text{if } q = 2 \\ 24p^2 + 32p + 11 & \text{if } q = 4 \\ 24p^2 + 44p + 20 & \text{if } q = 6 \end{cases}$$

Proof Choose a vertex v of S_t^* and its 6 closest vertices such that 4 of them are at distance $t+1$ and the remaining 2 are at distance $t+2$. By Lemma 2, each of these vertices can be perceived as a center of a diamond. The vertices of each diamond must belong all to different independent sets because they are pairwise at distance at most t. Building such a diamond around each vertex of S_t^*, one obtains a tessellation of H with some *uncovered* vertices between any two diamonds whose centers are at distance $t+2$. A possible placement of the vertices of the maximum t-independent set is depicted in Figure 7 for $t = 8$ (on the left) and $t = 10$ (on the right). Note that there is no way to decrease the number of uncovered vertices because, by Lemmas 4 and 5, v and its 6 closest vertices are as dense as possible. Clearly, these uncovered vertices cannot belong to S_t^*.

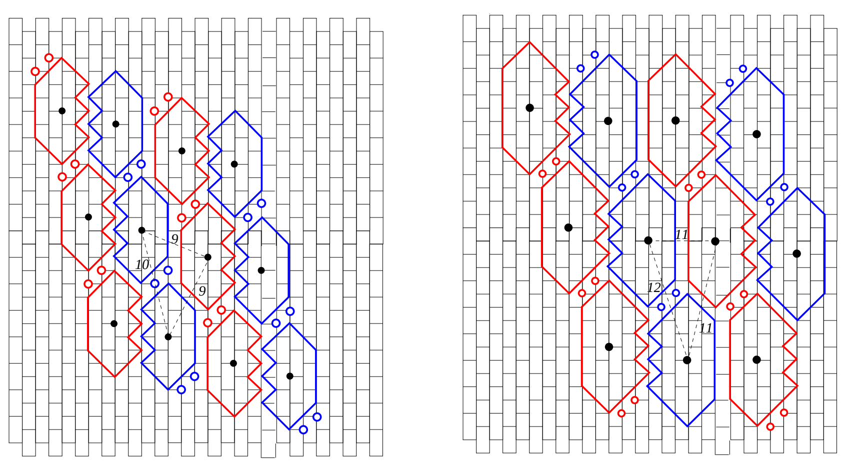

Fig. 7. The maximum 8-independent set (left) and 10-independent set (right) consisting of the diamond centers (depicted by black dots). The uncovered vertices are shown by white circles.

Since every center has two closest vertices at distance $t+2$, by observing Figure 7, one notes that there are two groups of uncovered vertices adjacent to any diamond: one group is above the diamond left columns and the other group is below the diamond right columns. Now, a mapping can be defined between each diamond center and the uncovered vertices by assigning to a diamond with a left center the uncovered vertices above its left columns, and assigning to a diamond with a right center the uncovered vertices above its right columns. Note that if the two diamond centers are at distance $t+2$, they are both either left or right centers. Hence, the mapping assigns the uncovered vertices between the two diamonds to one and only one of them.

Finally, the number of uncovered vertices associated to each diamond is the minimum between the number of left and right columns of the diamond, which in turn is exactly equal to the number h of holes of the diamond, that is: $h = 2p$ if $q = 0, 2$, or $h = 2p + 1$ if $q = 4, 6$. Hence, only one vertex out of the diamond vertices and its holes, that is one out of $\omega(A_{H,t}) + h$ vertices, can belong to S_t^*. Therefore, $\mu_t(H) \geq \omega(A_{H,t}) + h$, and by Lemma 2 the proof follows. ∇

Optimal coloring By Lemma 6, to derive optimal colorings when t is even, one only needs to consider a diamond enlarged in such a way that it includes also all its h holes. Then one tessellates the grid by means of the enlarged diamonds, using exactly the same techniques already seen in the case that t is odd.

4 Conclusion

Table 3 summarizes the results for optimal $L(1^t)$-coloring for the three grids based on regular tessellations. By observing Table 3, one notes that the proposed colorings for honeycomb grids use less colors than those required by the hexagonal and square grids for any $t > 1$. Therefore, honeycomb grids beat the hexagonal and square grids in terms of both network cost and channel requirement. However further work has still to be done on honeycomb grids. For instance, one could study the $L(\delta_1, \delta_2)$- and $L(\delta_1, 1^{t-1})$-coloring problems with arbitrary δ_1, δ_2 or t.

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Network G	$L(1^t)$
honeycomb grid	$\chi_t = \begin{cases} 24p^2+8p+1 & \text{if } q=0 \\ 24p^2+12p+2 & \text{if } q=1 \\ 24p^2+20p+4 & \text{if } q=2 \\ 24p^2+24p+6 & \text{if } q=3 \\ 24p^2+32p+11 & \text{if } q=4 \\ 24p^2+36p+14 & \text{if } q=5 \\ 24p^2+44p+20 & \text{if } q=6 \\ 24p^2+48p+24 & \text{if } q=7 \end{cases}$
square grid	$\chi_t = \lceil \frac{(t+1)^2}{2} \rceil$
hexagonal grid	$\lceil \frac{3}{4}(t+1)^2 \rceil \leq \chi_t \leq (t+1)^2 - t$
`References`	[1, 12, 13]

Table 3. Minimum number χ_t of channels used for a sufficiently large network G (for honeycomb grids, $t = 8p + q$, with $p \geq 0$ and $0 \leq q \leq 7$).

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