Root Lattices are Efficiently Generated

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Abstract

We show that root lattices are exactly those lattices whose relevant vectors are all equal in length. We define notions of "compatibility" between the euclidean metric and the shortest path metric in the infinite directed graph induced by a subset, G, of a lattice, i.e., the directed graph whose vertices are the points in the lattice, and whose arcs are the ordered pairs $(x, x+g)$, with x a lattice point and g a point in G. We present some ("easy to check for") criteria that a lattice and a subset of it must satisfy to ensure "compatibility" between the corresponding graphical and the euclidean metrics. We use these criteria to characterize, in more than one way, a set of "economically and efficiently generated" lattices, including root lattices. Our results include a "graph theoretic" characterization of root lattices as well. We also discuss, in brief, certain algorithmic considerations in the simulation of macroscopic physical phenomena in massively parallel computers based on suitable discretizations of euclidean space that led us to our graphical treatment of lattices.

1 Introduction

For any lattice $\mathcal{L}$, in euclidean space, each subset G of vectors in $\mathcal{L}$ induces an infinite directed graph, $\mathcal{G} = \mathcal{G}(\mathcal{L}, G)$, in the following way. The vertices of $\mathcal{G}$ are the points in $\mathcal{L}$, and the directed edges are the ordered pairs $(x, x+g), x \in \mathcal{L}, g \in G$. Clearly, $\mathcal{G}$ is strongly connected iff G is a *positive generating set* for $\mathcal{L}$, i.e., $\mathcal{L}$ is the set of all linear combinations over the natural numbers of vectors in G. For a fixed $\mathcal{G}(\mathcal{L}, G)$, $\mathcal{L}$ can be considered as a metric space in two natural ways: in the usual way, with respect to the euclidean metric d_e, and with respect to the graphical path length metric d_G. The two metric spaces can never coincide, and yet, often, properties of the one depend on those of the other. In this paper, we study how $G \subseteq \mathcal{L}$ determines properties of the metric space $(\mathcal{L}, d_G)$, and how these, in turn, determine properties of $(\mathcal{L}, d_e)$. In particular, we are interested in characterizing those subsets G of $\mathcal{L}$ such that d_G resembles d_e. We introduce two reasonable criteria to qualitatively measure such resemblance.

The first is the notion of d_G being *locally compatible* with the euclidean metric. We show that this notion is fairly natural and essential in algorithms for analogically simulating physical phenomena in cellular automaton-like models of computation (see Section 7). Intuitively, d_G is locally compatible with the euclidean metric exactly if $\mathcal{G}$ is strongly connected, and *locally*, d_e is a monotone increasing function of d_G. In particular, if d_G is locally compatible with the euclidean metric, then G must be a positive generating set for $\mathcal{L}$. Actually, in Section 2, we first define the function $\| \ \|_G$ on points $p \in \mathcal{L}$, by $\|p\|_G = d_G(\vec{0}, p)$. Then, we define instead, the local compatibility of $\| \ \|_G$ with the euclidean norm $\| \ \|$, associated with d_e in a similar way. Indeed, this is equivalent to defining the local compatibility of d_G because d_G is *translation invariant*, i.e., for $p, q \in \mathcal{L}$, $d_G(x, y) = d_G(\vec{0}, y - x)$.

A shortest path in $\mathcal{G}$ is a sequence of edges (straight line segments), whereas the shortest path in euclidean space is simply a single straight line segment. Our second criterion compares the shortest paths in $\mathcal{G}$ with the corresponding shortest paths in euclidean space. For this we introduce the notion of *intrinsically increasing paths* in $\mathcal{G}$ (see Section 2). Intuitively, stepping across successive edges in an intrinsically increasing path does not take one further away (in the euclidean distance sense) from the end point of the path. So, ideally, we would like all shortest paths in $\mathcal{G}(\mathcal{L}, G)$ to be intrinsically increasing. This is so iff G satisfies the easily verifiable property of being an *efficient* set (see Section 3). We also show that if G positively generates $\mathcal{L}$, then, efficiency of G ensures that d_G is locally compatible with d_e.

In Section 4 we study the set of relevant vectors in a lattice with respect to our notion of efficiency, and show that every efficient positive generating set for a lattice must contain every relevant vector in that lattice. In Section 5 we introduce two more notions for positive generating sets for lattices: *economically efficient* sets and *radial* sets. Radial sets allow one to recursively extend shortest paths, using only *local* information about relative angles between vectors. We then show that these two notions are equivalent. Not every lattice contains radial sets. In fact, we show that the only lattices that do contain a radial set are those for which the set of relevant vectors is efficient, and coincidentally, is the unique radial set in the lattice.

In Section 6 we concentrate on *minimally generated* lattices, i.e., lattices for which the set of minimal length vectors is a positive generating set. We show that if $\mathcal{L}$ is such a lattice, and G is the set of minimal length vectors of $\mathcal{L}$, G is efficient iff d_G is locally compatible with the euclidean metric. Furthermore, in a minimally generated lattice, the set of minmal length vectors is efficient, iff it is economically so, and therefore is radial.

The set of minimal length vectors in any lattice is contained in the set of relevant vectors of that lattice, and the containment could, in general, be proper. Theorem 9 on page 474 of [3], gives a sufficient condition for the set of minimal length vectors in a lattice to be the set of relevant vectors in that lattice. We call this condition the *Shell Condition*[1]. The Shell Condition, in effect, compares the metrics d_e, and d_G with G being the set of minimal length vectors. We show that in a minimally generated lattice, the Shell Condition is satisfied iff the set of minimal length vectors in the lattice is efficient, which in turn is true iff the lattice is a *root* lattice. Thus we show that the Shell Condition is both necessary and sufficient for a lattice to be a root lattice.

In our concluding theorem of this paper, we draw upon all the machinery developed in the paper and present nine different characterizations of root lattices.

2 Basic Definitions, Notation, and Examples

Definition 1 *Suppose $\mathcal{L}$ is a (real) n-dimensional lattice and G is a subset of vectors in $\mathcal{L}$.*

1. *$\| \ \|$ denotes the euclidean norm. $\langle G\rangle$ and $\langle G\rangle_{\mathbf{R}}$ denote the set of linear combinations over the natural numbers and over reals, respectively, of elements of G. A point $p \in \mathcal{L}$ is called a G-neighbor of a point $q \in \mathcal{L}$ iff $p = q + g$ for some $g \in G$. G is a* positive generating set for $\mathcal{L}$ *iff $\langle G\rangle = \mathcal{L}$. G is* weakly acute *iff the inner product of any pair of vectors in G is non-negative. G is* balanced *iff for every $g \in G$, $-g \in G$.* Note: Positive generating sets are never weakly acute, and need not be balanced..

[1]This name was suggested to us by John Conway and Neil Sloane.

2. *A G-*path *is a finite sequence of elements of G. The empty G-path is denoted $[\ ]$. $[g_1, \ldots, g_k]$, with $g_1, \ldots, g_k \in G$, denotes a G-path. Each element in a G-path is called an* edge *of that G-path. If $P = [g_1, \ldots, g_k]$ is an G-path, and $v \in G$, then $(P \circ v)$ denotes the G-path $[g_1, \ldots, g_k, v]$. The* endpoint of a G-path, *$P = [g_1, \ldots, g_k]$, is the (vector) sum, $\sum_{i=1}^{k} g_i$, of all the edges in P.* Note: We consider the G-path P to begin at the origin of $\mathcal{L}$ and end at the endpoint of P. *The* underlying set *of a G-path P, denoted $\mathbf{U}(P)$, is the set of (pairwise distinct) vectors occurring in P. A G-path is said to be* intrinsically increasing *iff the underlying set of that path is weakly acute. A G-*subpath *of a G-path, P, is a subsequence of consecutively occurring edges in P.*

3. *The* graphical path length *of a G-path, P, is the number of edges in P, and is denoted by $GPL_G(P)$. A G-path, P, is a* shortest G-path *iff every G-path with the same endpoint as that of P has graphical path length at least $GPL_G(P)$. If P is a shortest G-path and $v \in G$, v* extends *P iff $P \circ v$ is also a shortest G-path. For any $p \in \mathcal{L}$, $\mathbf{S}_G(p)$ denotes the set of all shortest G-paths with endpoint p, and $\mathbf{C}_G(p)$ denotes the set $\cup_{P \in \mathbf{S}_G(p)} \mathbf{U}(P)$. For any point $p \in \mathcal{L}$, the* graphical distance of p with respect to G, *denoted $\|p\|_G$, is the path length of a shortest G-path with endpoint p. If there is no G-path to p, then we define $\|p\|_G$ to be ∞.* Note: $\langle G \rangle = \{p \in \mathcal{L} \mid \|p\|_G < \infty\}$..

4. *$\|\ \|_G$ is* locally compatible *with the euclidean norm iff for every $p \in \mathcal{L}$, $\|p\|_G < \infty$ and for every G-neighbor $q \in \mathcal{L}$ of p, with $\|p\|_G < \|q\|_G$, $\|p\| < \|q\|$.*

Example 1 Let $\mathcal{L}$ be a lattice with its origin denoted by $\vec{0}$, and let G be a subset of $\mathcal{L}$. The empty sequence is a G-path. The empty path is an intrinsically increasing G-path, a shortest G-path with endpoint $\vec{0}$ and graphical path length 0, and is a G-subpath of every G-path. If $\vec{0} \in G$, then $[\vec{0}]$ is an intrinsically increasing G-path, but not a shortest G-path.

Example 2
Let $\mathcal{L} = \mathbf{Z}^2$. $G = \{g_1, g_2, g_3\}$, shown in the diagram on the left hand side of Figure 1, is not balanced, is positive generating for $\mathbf{Z}^2$, and is not weakly acute. The subset $\{g_1, g_2\}$ of G is weakly acute, but not positive generating for $\mathbf{Z}^2$. The diagram on the right hand side of Figure 1 shows two G-paths, $P = [g_1, g_1, g_2]$, and $Q = [g_3, g_3, g_1, g_1]$. P is an intrinsically increasing G-path, whereas Q is not. $\sum P = p$, and $\sum Q = q$. Note that, for this G, every G-path is a shortest path. Also, $\|p\|_G = 3$, and $\|q\|_G = 4$. $\mathbf{U}(P) = \{g_1, g_2\} = \mathbf{C}_G(p)$, $\mathbf{U}(Q) = \{g_1, g_3\} = \mathbf{C}_G(q)$, and $\mathbf{S}_G(p) = \{[g_1, g_1, g_2], [g_1, g_2, g_1], [g_2, g_1, g_1]\}$. Note that not all shortest G-paths are intrinsically increasing.

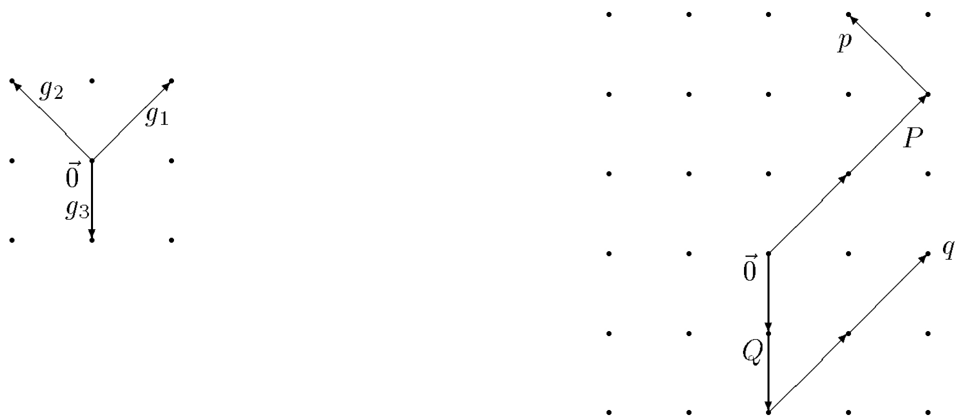

Figure 1: G and G-paths

Example 3

Let $\mathcal{L} = \mathbf{Z}^2$. The diagram on the left in Figure 2 shows a set, $H = \{h_i \mid 1 \le i \le 8\}$, of vectors in $\mathbf{Z}^2$, and the diagram on the right shows four H-paths, P, Q, R, and S, all having the same endpoint, q. H is a balanced positive generating set for $\mathbf{Z}^2$, and it is not weakly acute. $\mathbf{U}(P) = \{h_1, h_2, h_7\}$, $\mathbf{U}(Q) = \mathbf{U}(S) = \{h_2, h_8\}$, and $\mathbf{U}(R) = \{h_1\}$. The H-path P is not a shortest path while the other three are. So, $\mathbf{S}_H(q) = \{Q, R, S\}$, and $\mathbf{C}_H(q) = \{h_1, h_2, h_8\}$. Note that, every shortest H-path is intrinsically increasing.

3 Intrinsically Increasing G-Paths and Efficient Sets

Henceforth, $\mathcal{L}$ will always denote a real, n-dimensional lattice, G will denote a subset of vectors in $\mathcal{L}$, and $\vec{0}$ will denote the origin of $\mathcal{L}$. We start this section with the following observations which will be used implicitly through most of this paper.

Lemma 1

1. *G-subpaths of shortest G-paths are shortest G-paths.*
2. *$[g_1, \ldots, g_k]$ is a shortest G-path, iff for every permutation, σ, of degree k, $[g_{\sigma(1)}, \ldots, g_{\sigma(k)}]$ is a shortest G-path in $\mathcal{L}$.*
3. *In any intrinsically increasing path, P, with end-point p, if g is a vector in $\mathbf{U}(P)$, $(p - g, g) \ge 0$.*

$\square$

Figure 2: H and H-paths

Definition 2 *G is* efficient*, iff the non-zero sum of every pair of vectors in G, whose inner-product is negative, is in G. The* efficient closure *of a set G is the set, denoted by $\overline{G}$, which is the intersection of all efficient supersets of G.*

Example 4 The set of unit length vectors in $\mathbf{Z}^n$ is efficient. H, of Example 3, is efficient, but G of Example 2 is not.

Theorem 1 *The following are equivalent.*

1. *Every shortest path of vectors from G is intrinsically increasing.*
2. *Every shortest path of length 2 of vectors from G is intrinsically increasing.*
3. *G is efficient.*

Proof. (1 *iff* 2): The statement is trivially true if G is empty or contains a single vector. So let G contain at least two vectors.

Suppose every shortest G-path of length 2 is an intrinsically increasing G-path. If, by way of contradiction, the inner product of some two vectors in some shortest G-path is negative, then by Lemma 1(2), we can assume that these are the first two edges in that G-path. Then, by Lemma 1(1), these two edges form a shortest G-path which is not intrinsically increasing.

The converse is obvious.

(2 *iff* 3): The statement is trivially true if G is empty, or contains a single vector. The proposition is also trivially true if the inner product of every pair of non-collinear vectors in G is 0. So let G contain at least two vectors whose inner product is non-zero.

Now, suppose every shortest G-path of length 2 is intrinsically increasing. Suppose G contains two vectors g, g' such that $(g, g') < 0$. Then, by our current hypothesis, $[g, g']$ cannot be a shortest G-path, and hence, G must contain the vector $(g + g')$. Thus G is efficient.

Conversely, suppose G is efficient. Then, any G-path of length 2 that is not intrinsically increasing, i.e., contains two vectors $g, g' \in G$ such that $(g, g') < 0$, is not a shortest path. increasing. $\square$

Theorem 2 *If G is an efficient positive generating set for $\mathcal{L}$, then $\| \, \|_G$ is locally compatible with the euclidean norm on $\mathcal{L}$.*

PROOF. Suppose G is an efficient positive generating set for $\mathcal{L}$. Let $p, q \in \mathcal{L}$ with $q = p + g$, $g \in G$, and $\|p\|_G < \|q\|_G$. Then, $\|q\|_G = \|p\|_G + 1$. Since G is efficient, every shortest path is increasing, and hence $(p, g) \geq 0$. Therefore,

$$\|q\|^2 = \|p\|^2 + 2(p, g) + \|g\|^2 > \|p\|^2.$$

$\square$

4 Relevant Vectors

In this section, we consider positive generating, efficient $G \subseteq \mathcal{L}$. In Theorem 3 below, we show that the set of relevant vectors of $\mathcal{L}$ (see Definition 3, below) is contained in every such G. The Voronoi cell, around $\vec{0}$, of $\mathcal{L}$, denoted $V(\vec{0}, \mathcal{L})$, is the set of all points in $\langle \mathcal{L} \rangle_{\mathbf{R}}$ which are at least as close to the origin as to any other point in $\mathcal{L}$. See [3] for the usual definitions of relevant vectors of a lattice, and the Voronoi cell around the origin of the lattice $\mathcal{L}$. We will use the following characterization, due to Voronoi, of relevant vectors as our definition of a relevant vector (see [3, pp. 474-5]).

Definition 3 (Voronoi's Characterization) *r is a* relevant vector of $\mathcal{L}$ *iff for every $p \in (\mathcal{L} \setminus \{\vec{0}, r\})$, $(r - 2p, r - 2p) > (r, r)$. Rel($\mathcal{L}$) will denote the set of relevant vectors in $\mathcal{L}$.*

Proposition 1 *Rel($\mathcal{L}$) is a balanced positive generating set for $\mathcal{L}$.* $\square$

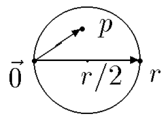

Figure 3: Relevant and Non-Relevant vectors

The following corollaries to Voronoi's characterization are especially useful in our later proofs.

Corollary 1

1. *A point r in $\mathcal{L}$ is a relevant vector iff for every nonzero vector, $p \neq r$, in $\mathcal{L}$, $(p, p) > (p, r)$.*
2. *A point p in $\mathcal{L}$ is not a relevant vector of $\mathcal{L}$ iff there is a relevant vector, $r \neq p$, of $\mathcal{L}$ so that $(p - r, r) \geq 0$.*
3. *r is a relevant vector of $\mathcal{L}$ iff $[r]$ is the only intrinsically increasing $\mathcal{L}$-path with endpoint r.*

Proof. (1) is immediate from Definition 3 and (2) follows from Definition 3 and Proposition 1. We prove (3).

By (1) above, r is a relevant vector of $\mathcal{L}$ iff for every nonzero vector, $p \neq r$, in $\mathcal{L}$, $(p, r - p) < 0$. Then, by the contra-positive of Lemma 1(3), r is a relevant vector of $\mathcal{L}$ iff for every nonzero vector, $p \neq r$, in $\mathcal{L}$, p is not on an intrinsically increasing path whose end point is r. $\square$

Figure 3 gives a geometric interpretation of Corollary 1(1). Essentially, if r is a relevant vector of $\mathcal{L}$, then, there is no point of $\mathcal{L}$ that lies inside, or on the surface of, the n-dimensional sphere of radius half the length of r and centered at $r/2$. Equivalently, as pointed out in Corollary 1(2) below, for every vector r in $\mathcal{L}$ that is not a relevant vector of $\mathcal{L}$, there is a relevant vector (shown as p in Figure 3) inside, or on the surface of, such an n-dimensional sphere centered at $p/2$.

Theorem 3 *If G is an efficient, positive generating set for $\mathcal{L}$, then G contains $\overline{Rel(\mathcal{L})}$. That is, $\overline{Rel(\mathcal{L})}$ is the minimal efficient positive generating set for $\mathcal{L}$.*

PROOF. Since G is an efficient, positive generating set for $\mathcal{L}$, by Theorem 1, for each relevant vector, r, of $\mathcal{L}$, there is an intrinsically increasing shortest G-path with endpoint r. By Corollary 1(3) this path must be $[r]$, so $r \in G$. Since $\overline{\mathrm{Rel}(\mathcal{L})}$ is the smallest set which is efficient and contains $\mathrm{Rel}(\mathcal{L})$, we have $\overline{\mathrm{Rel}(\mathcal{L})} \subseteq G$. $\square$

Corollary 2

1. *If G is an efficient positive generating set for $\mathcal{L}$, then G contains a balanced, efficient positive generating set for $\mathcal{L}$.*
2. *If $G \subseteq Rel(\mathcal{L})$ is a positive generating set for $\mathcal{L}$, then $G \subseteq Rel(\mathcal{L}) \subseteq \overline{G} = \overline{Rel(\mathcal{L})}$.*
3. *If $G \subseteq Rel(\mathcal{L})$ is an efficient, positive generating set for $\mathcal{L}$, then $G = Rel(\mathcal{L}) = \overline{G} = \overline{Rel(\mathcal{L})}$.* $\square$

Example 5 $\mathrm{Rel}(\mathcal{L}) = \overline{\mathrm{Rel}(\mathcal{L})}$ in ${A_n}^*$, E_6^*, and E_7^*, and in any root lattice. (See Definition 7 below and [3] for details on root lattices.)

Example 6 For several lattices, $\mathrm{Rel}(\mathcal{L}) \neq \overline{\mathrm{Rel}(\mathcal{L})}$.

1. Let $a, b, c \in \mathbf{Z}^7$ be defined as follows.

$$\begin{aligned} a &= (1,1,1,1,1,1,0), \\ b &= (-1,-1,-1,-1,1,1,0), \text{ and} \\ c &= (0,0,0,0,1,1,1). \end{aligned}$$

   Let $\mathcal{L}$ be the (3-dimensional) sublattice of $\mathbf{Z}^7$ generated by $\{a, b, c\}$. a, b, c are relevant vectors of $\mathcal{L}$. Table 1 gives the inner products of eight vectors in $\mathcal{L}$, which, together with their negatives, form an efficient set. By Corollary 2(2), this set is $\overline{\mathrm{Rel}(\mathcal{L})}$. All the vectors shown in the table, except $(a+b)$, are relevant vectors of $\mathcal{L}$. So $\mathrm{Rel}(\mathcal{L}) = \overline{\mathrm{Rel}(\mathcal{L})} \setminus \{\pm(a+b)\}$.

2. For $n \geq 5$, $\mathrm{Rel}(D_n^*) \neq \overline{\mathrm{Rel}(D_n^*)}$ (See [3] for a definition of D_n^*).

5 Economically Efficient Sets and Radiating Sets

Definition 4 *G is* economically efficient *iff G is efficient and, if, for all $a, b \in G$, $(a+b) \in G$ implies $(a, b) < 0$.*

	a	b	c	$(a+b)$	$(a-c)$	$(b-c)$	$(a+b-c)$	$(a+b-2c)$
a	6	-2	2	4	4	-4	2	0
b	-2	6	2	4	-4	4	2	0
c	2	2	3	4	-1	-1	1	-2
$(a+b)$	4	4	4	8	0	0	4	0
$(a-c)$	4	-4	-1	0	5	-3	1	2
$(b-c)$	-4	4	-1	0	-3	5	1	2
$(a+b-c)$	2	2	1	4	1	1	3	2
$(a+b-2c)$	0	0	-2	0	2	2	2	4

Table 1: Innerproducts of vectors in $\overline{\mathrm{Rel}(\mathcal{L})}$.

Example 7 $\emptyset$ is always economically efficient, and so is any $\{v\} \subseteq \mathcal{L}$, $v \neq \vec{0}$. The set of minimal length vectors in $\mathbf{Z}^n$ and A_n are economically efficient. G, from Example 2, is not efficient, and hence not economically efficient. H of Example 3 is efficient, but not economically efficient since, $(h_1, h_3) \not< 0$ but $h_2 = (h_1 + h_3) \in H$. The set $\overline{\mathrm{Rel}(\mathcal{L})}$ in Example 6(1) is not economically efficient.

Lemma 2

1. *If G is economically efficient, then $\vec{0} \notin G$.*

2. *Every efficient subset of $Rel(\mathcal{L})$ is economically efficient.*

Proof. (1) is trivial, and (2) follows from Corollary 1(3). $\square$

Proposition 2 *G is economically efficient iff the set of intrinsically increasing G-paths of length 2 is the same as the set of shortest G-paths of length 2.*

Proof. As a consequence of Theorem 1, it suffices to show that, for all efficient G, every intrinsically increasing G-path of length 2 is shortest iff G is economically efficient. If G is efficient, then, G is not economically efficient iff there are $a, b \in G$ so that $(a+b) \in G$ and $(a, b) \geq 0$, i.e., so that $[a, b]$ is an intrinsically increasing G-path of length 2 which is not shortest. $\square$

The following proposition shows that vectors in economically efficient, balanced sets have a property similar to one, described in Corollary 1(3), of relevant vectors of a lattice.

Proposition 3 *Let G be a balanced set of vectors. Then, G is economically efficient iff G is efficient, and for all $g \in G$, the only intrinsically increasing G-path with endpoint g, is $[g]$.*

PROOF. Suppose G is efficient and for all $g \in G$, the only intrinsically increasing G-path with endpoint g, is $[g]$. Then, in particular, for all $\vec{0} \neq g \in G$, there is no intrinsically increasing G-path of length 2, with endpoint g. That is, every intrinsically increasing G-path of length 2 is shortest. Then, by Proposition 2, G is economically efficient.

Conversely, suppose G is economically efficient. Then, G is, by definition, efficient. Suppose, by way of contradiction, $\vec{0} \neq g \in G$ so that there is an intrinsically increasing G-path, $P \neq [g]$, with endpoint g. Then, by Lemma 1(3),

$$\text{for all } p \in \mathbf{U}(P), (g-p, p) \geq 0. \tag{1}$$

CLAIM. For every $p \in \mathbf{U}(P)$, $(p, g) = 0$.
PROOF OF CLAIM. Suppose, by way of contradiction, there is a $p \in \mathbf{U}(P)$ so that $(p, g) > 0$. Then $(-p, g) < 0$, and so $(g-p) \in G$ (since G is efficient and balanced). Then, by Equation (1), $[g-p, p]$ is an intrinsically increasing G-path of length 2, but not a shortest G-path (since G contains g). Thus, by Proposition 2, G is not economically efficient, giving us a contradiction. □ CLAIM

But now, $(g, g) = (g, \sum_{p \in P} p) = 0$, i.e., $g = \vec{0} \in G$. Thus, by Lemma 2 (1), G is not economically efficient, giving us a contradiction. □

Definition 5 *G is* radial *iff for every $p \in \langle G \rangle$, and every $g \in G$, g extends every element of $\mathbf{S}_G(p)$ iff $\{g\} \cup \mathbf{C}_G(p)$ is weakly acute.*

Example 8 In the example shown in Figure 2 above, h_1, h_2, and h_8 each extend Q, R, and S. Let $p = (0, 1)$. Then, p is the endpoint of the H-path $[h_1]$, a shortest H-path. $\{h_3\} \cup \mathbf{C}_H(p)$ is weakly acute, but h_3 does not extend $[h_1]$. So, H is not a radial set.

Proposition 4 *A balanced set of vectors, G, is radial iff G is economically efficient.*

PROOF. Let G be balanced. Suppose G is radial. Suppose $[g, g']$ is an intrinsically increasing G-path. Then, $\{g'\} \cup \mathbf{C}_G(g)$ is weakly acute, and so g' extends the shortest G-path $[g]$, i.e., $[g, g']$ is a shortest G-path. Thus, every intrinsically increasing G-path of length 2 is a shortest G-path. Also, if $[g, g']$ is a shortest G-path, then, since G is radial, g', g is weakly acute, i.e., $[g, g']$ is intrinsically increasing. So, by Proposition 2, G is economically efficient.

To prove the converse, we need to show that for every $p \in \langle G \rangle$, and every $g \in G$, g extends every element of $\mathbf{S}_G(p)$ iff $\{g\} \cup \mathbf{C}_G(p)$ is weakly acute. To prove the implication in the forward direction, suppose G is economically efficient and let $p \in \langle G \rangle$. If a vector, $g \in G$, extends every shortest G-path, P, to p, then by Definition 5, for all such P, $P \circ g$ is a shortest G-path and so must be intrinsically increasing by Theorem 1. So, $\{g\} \cup \mathbf{C}_G(p)$ is weakly acute.

It remains to be shown that for every $p \in \langle G \rangle$, and every $g \in G$, if $\{g\} \cup \mathbf{C}_G(p)$ is weakly acute, then g extends every element of $\mathbf{S}_G(p)$. So, suppose $p \in \langle G \rangle$ with $\|p\|_G = k$. If $k = 0$, $\mathbf{S}_G(p) = \{[\,]\}$, and so, the result follows trivially. So, assume that $k > 0$. Let $g \in G$ and let $\mathbf{C}_G(p) \cup \{g\}$ be weakly acute. Let $P \in \mathbf{S}_G(p)$ and let $Q \in \mathbf{S}_G(p+g)$. Then, $P \circ g$ is intrinsically increasing. Also, since G is balanced, $-g \in G$, so $Q \circ (-g)$ is a G-path with endpoint p. Now, since $\|p\|_G = k$, $\mathrm{GPL}_G(Q \circ (-g)) \geq k$, and by Lemma 1(1), $\mathrm{GPL}_G(Q) \geq (k-1)$. We will show that $\mathrm{GPL}_G(Q) > k$, thus proving that g extends the shortest path P. (See Figure 4 below.)

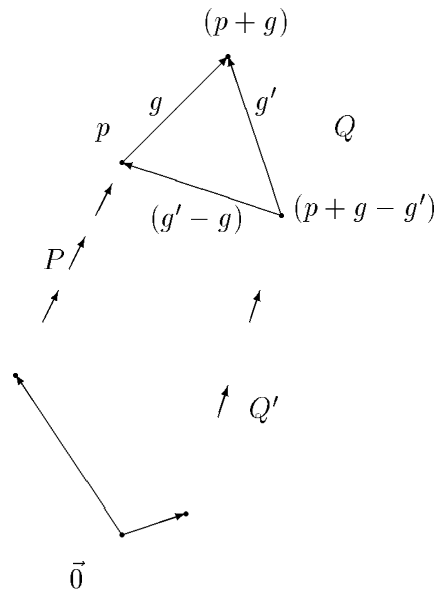

Figure 4: g properly extends P

If $\mathrm{GPL}_G(Q) = (k-1)$, $Q \circ (-g)$ is a shortest G-path to p, and so since G is efficient, by Theorem 1, for every $g' \in \mathbf{U}(Q)$, $(-g, g') \geq 0$. By our choice of g, $(g, g') \geq 0$. Also, since G is economically efficient, by Lemma 2(1), $g \neq \vec{0}$, the only vector which is orthogonal to both g and $-g$ is $\vec{0}$. But, again by Lemma 2(1), if g' exists, then $g' \neq \vec{0}$. So we must have $\mathbf{U}(Q) = \emptyset$, so that $Q \circ (-g) = [-g] = [p]$. But then, $(g, -g) < 0$, which contradicts our choice of g.

If $\mathrm{GPL}_G(Q) = k$, then, $k > 1$ by Proposition 2 and 3. Now, since $\{g\} \cup \mathbf{C}_G(p)$ is weakly acute, $(g, \sum P) = (g, p) \geq 0$. Since $g \neq \vec{0}$, $(g, p+g) > 0$ and so, there is a g' in $\mathbf{U}(Q)$ with $(g, g') > 0$. Let Q' be the G-subpath of Q, with endpoint $p+g-g'$. By Lemma 1(1) $\mathrm{GPL}_G(Q') = (k-1)$. So, we must have $g \neq g'$, for assuming otherwise would lead to the contradiction that $\|p\|_G = \|p+g-g'\|_G = k-1$. Now, since G is efficient, $(g'-g) \in G$ so that $Q' \circ (g'-g)$ is a shortest G-path with endpoint p. i.e., $Q' \circ (g'-g) \in \mathbf{S}_G(p)$. Our choice of g implies that $(g, (g'-g)) \geq 0$, so that $[g, (g'-g)]$ is an intrinsically increasing G-path with endpoint g'. Again this contradicts Proposition 3. $\square$

As the following example shows, in the definition of a radial set, it is essential that $g \in G$ extend *every* element of $\mathbf{S}_G(p)$.

Example 9 If G is the set of minimal length vectors in D_4, then, G is a balanced economically efficient set, and

$$P = [(2,0,0,0), (0,2,0,0)]$$

is a shortest G-path to $p = (2,2,0,0)$. Consider the vector $g = (0,0,2,0)$. Then, $P \circ g$ is intrinsically increasing, but g does not extend P since

$$[(1,1,1,1), (1,1,1,-1)]$$

is a G-path with the same endpoint as $P \circ g$. $\{g\} \cup \mathbf{C}_G(p)$ is not weakly acute because $(1,1,-1,-1) \in \mathbf{C}_G(p)$.

Proposition 5 *G is an economically efficient, positive generating set for $\mathcal{L}$ iff $G = Rel(\mathcal{L}) = \overline{Rel(\mathcal{L})}$.*

Proof. Suppose G is an economically efficient, positive generating set for $\mathcal{L}$. Then, G is efficient by definition and, by Theorem 3, $\mathrm{Rel}(\mathcal{L}) \subseteq \overline{\mathrm{Rel}(\mathcal{L})} \subseteq G$. Now, it suffices to show $G \subseteq \mathrm{Rel}(\mathcal{L})$. Suppose $p \in \mathcal{L} \setminus \mathrm{Rel}(\mathcal{L})$. Then, by Corollary 1(2), for some $r \in \mathrm{Rel}(\mathcal{L})$,

$$((p-r), r) \geq 0. \tag{2}$$

Since $\mathrm{Rel}(\mathcal{L})$ is balanced (Proposition 1), $(-r) \in \mathrm{Rel}(\mathcal{L})$, and from Equation (2) we have

$$(p, r) > 0. \tag{3}$$

Suppose, by way of contradiction, $p \in G$. Then, using Equation (3) and the facts that G is efficient and $(-r) \in \mathrm{Rel}(\mathcal{L})$, we have that

$$(p - r) \in G. \tag{4}$$

Then, $p = ((p - r) + r) \in G$, $r \in \mathrm{Rel}(\mathcal{L})$, and Equations (4) and (2) imply that G is not economically efficient, giving us a contradiction.

The converse follows from Lemmas 1 and 2(2). $\square$

Note: The condition $\mathrm{Rel}(\mathcal{L}) = \overline{\mathrm{Rel}(\mathcal{L})} = G$ is much stronger than the condition $G = \overline{\mathrm{Rel}(\mathcal{L})}$. Note that by Proposition 1, if G is an economically efficient positive generating set for $\mathcal{L}$, then G is balanced.

We collect the main results of this section in the following theorem.

Theorem 4 *If G is a positive generating set for $\mathcal{L}$, then the following are equivalent.*

1. *G is economically efficient.*

2. *G is balanced and economically efficient.*

3. *A G-path of length 2 is a shortest G-path iff that path is intrinsically increasing.*

4. *G is efficient and, for all $g \in G$, the only intrinsically increasing G-path with endpoint g, is $[g]$.*

5. *G is radial.*

6. *$Rel(\mathcal{L}) = \overline{Rel(\mathcal{L})} = G$.*

6 Minimal Vectors and Root Lattices

Every lattice, $\mathcal{L}$, has vectors of a characteristic, to $\mathcal{L}$, minimal length, which we will denote by $\mu(\mathcal{L})$. Also, we will denote the set of all minimal length vectors of $\mathcal{L}$ by $\mathrm{Min}(\mathcal{L})$. $\mathrm{Min}(\mathcal{L})$ is a balanced set.

Definition 6 *$\mathcal{L}$ is* minimally generated *iff $Min(\mathcal{L})$ is a (positive) generating set for $\mathcal{L}$.*

Section 2 in Chapter 4 of [3] gives a definition of a root lattice. For our purposes the following, equivalent definition is more useful. (The equivalence follows from Table 4.1 on page 98 in [3] and from the facts that (1) each minimal length vector is a root vector of $\mathcal{L}$, and hence (2) the lattice is generated by the roots of minimal length.)

Definition 7 *$\mathcal{L}$ is a root lattice iff it is minimally generated and the angle between any two distinct minimal length vectors in $\mathcal{L}$ is $k \cdot (\pi/t)$, for some integer k and $t \in \{2,3\}$.*

All root lattices are minimally generated as are A_n^*, E_6^* and E_7^*. For $n > 4$, D_n^* is not minimally generated.

Theorem 5 *Suppose $\mathcal{L}$ is minimally generated. Then, Min($\mathcal{L}$) is efficient iff $\| \; \|_G$ is locally compatible with the euclidean norm.*

Proof. By Theorem 2, we need to prove only that if $d\mathrm{Min}(\mathcal{L})$ is locally compatible with the euclidean norm, then $\mathrm{Min}(\mathcal{L})$ is efficient. Suppose that $d\mathrm{Min}(\mathcal{L})$ is locally compatible with the euclidean norm, and that $p, q \in \mathrm{Min}(\mathcal{L}), p+q \neq \vec{0}, (p,q) < 0$. Then, for $k \geq \mu(\mathcal{L})^2/(2|(p,q)|)$, $\|kp+q\| \leq k\|p\|$. Let $P = [g_1, \ldots, g_r]$ be a shortest path to $kp+q$. Then, $r = d\mathrm{Min}(\mathcal{L})(kp+q, \vec{0}) \leq d\mathrm{Min}(\mathcal{L})(kp, \vec{0}) = k$. Since $p, q, g_i, 1 \leq i \leq r$, are minimal length vectors, we have that

$$-\mu(\mathcal{L})/2 \leq (g_i, p), (p,q) \leq \mu(\mathcal{L})/2,$$

so that

$$(kp+q, p) = (\sum_{i=1}^{r} g_i, p) \leq r\mu(\mathcal{L})/2 \leq k\mu(\mathcal{L})/2,$$

and also

$$(kp+q, p) = k\mu(\mathcal{L}) + (p,q) \geq (k-1/2)\mu(\mathcal{L})/2.$$

We therefore have $k/2 \geq k - 1/2$, so, $k \leq 1$. By our choice of k, this implies that $|(p,q)| = \mu(\mathcal{L})/2$, so that in fact, $p+q \in \mathrm{Min}(\mathcal{L})$ and $\mathrm{Min}(\mathcal{L})$ is efficient. $\square$

Proposition 6

1. *$Min(\mathcal{L}) \subseteq Rel(\mathcal{L})$.*
2. *If $\mathcal{L}$ is minimally generated, then $\overline{Min(\mathcal{L})} = \overline{Rel(\mathcal{L})}$.*

PROOF. To prove (1), suppose $v \notin \mathrm{Rel}(\mathcal{L})$. We will show that v cannot be a minimal length vector. By Corollary 1(2), there is an $r \in \mathrm{Rel}(\mathcal{L})$, such that $(r, r) \leq (r, v)$. Clearly, r cannot be collinear with v, and so, $(r, v) < \|r\| \cdot \|v\|$. Therefore,

$$(r, r) \leq (r, v) < \|r\| \cdot \|v\| \leq (v, v),$$

implying that $v \notin \mathrm{Min}(\mathcal{L})$.

(2) follows immediately upon replacing $\mathrm{Min}(\mathcal{L})$ for G in Corollary 2 (2). $\square$

A_n^* with $n \geq 3$, and $\mathcal{L} = E_6^*$ or $\mathcal{L} = E_7^*$, are examples of lattices $\mathcal{L}$ in which $\mathrm{Min}(\mathcal{L}) \neq \mathrm{Rel}(\mathcal{L}) = \overline{\mathrm{Min}(\mathcal{L})} = \overline{\mathrm{Rel}(\mathcal{L})}$.

In general one would not expect that if $G = \mathrm{Rel}(\mathcal{L})$ then $G = \overline{G}$. So, the following theorem is interesting.

Theorem 6 *$Min(\mathcal{L}) = Rel(\mathcal{L})$ iff $\mathcal{L}$ is minimally generated and $Min(\mathcal{L})$ is efficient.*

PROOF. First, suppose $\mathrm{Min}(\mathcal{L}) = \mathrm{Rel}(\mathcal{L})$. Then, Proposition 1 implies that we need only show that $\mathrm{Min}(\mathcal{L})$ is efficient. Now, let $r_1 \neq r_2$ are relevant vectors so that $(r_1, r_2) < 0$, and let $r = (r_1 + r_2)$. Then,

$$(r, r) < (r_1, r_1) + (r_2, r_2). \tag{5}$$

Suppose, by way of contradiction, r is not relevant. Then, by Corollary 1(1), there exists a $p \in \mathcal{L}$ so that $(p, r) \geq (p, p)$, so that $(p, r - p) \geq 0$. Since r_1, r_2 are of minimal length,

$$\begin{aligned}
(r, r) &= (r - p, r - p) + 2(r - p, p) + (p, p), \\
&\geq (r - p, r - p) + (p, p), \\
&\geq (r_1, r_1) + (r_2, r_2),
\end{aligned}$$

contradicting (5) above.

The proof that $\mathrm{Min}(\mathcal{L}) = \mathrm{Rel}(\mathcal{L})$ if $\mathcal{L}$ is minimally generated and $\mathrm{Min}(\mathcal{L})$ is efficient, is immediate upon replacing $\mathrm{Min}(\mathcal{L})$ for G in Corollary 2(3). $\square$

Lemma 3 *Suppose $\mathcal{L}$ is minimally generated. Then, $Min(\mathcal{L})$ is efficient iff for all $p \neq p' \in Min(\mathcal{L})$, $(p, p')/\mu(\mathcal{L})^2 = 0, \pm 1/2$. i.e., iff the angle between any two distinct minimal length vectors in $\mathcal{L}$ is $k \cdot (\pi/t)$, for some integer k and $t \in \{2, 3\}$.*

PROOF.

$$\begin{aligned}
&\mathrm{Min}(\mathcal{L}) \text{ is efficient iff}\\
&(\forall p, p' \in \mathrm{Min}(\mathcal{L}) \mid (p,p') < 0)\ (p+p') \in \mathrm{Min}(\mathcal{L}) \text{ iff}\\
&(\forall p, p' \in \mathrm{Min}(\mathcal{L}) \mid (p,p') < 0)\\
&\qquad (p+p', p+p') = (p,p) + 2(p,p') + (p',p') = \mu(\mathcal{L})^2 \text{ iff}\\
&(\forall p, p' \in \mathrm{Min}(\mathcal{L}) \mid (p,p') < 0)\ 2(p,p') = -\mu(\mathcal{L})^2.
\end{aligned}$$

□

Corollary 3 *Suppose $\mathcal{L}$ is minimally generated. Then, if Min($\mathcal{L}$) is efficient then the norm of any vector in $\mathcal{L}$ is an integer multiple of the minimal norm of $\mathcal{L}$.*

PROOF. Since $\mathcal{L}$ is minimally generated, each point p in $\mathcal{L}$ is the endpoint of a Min($\mathcal{L}$)-path, $[v_1, \ldots, v_k]$. Then,

$$(p,p) = \left[\sum_{1 \le i \le k} (v_i, v_i)\right] + 2\left[\sum_{1 \le i < j \le k} (v_i, v_j)\right],$$

which, by Lemma 3, equals $\mu(\mathcal{L})^2 \cdot t$, for some integer t. □

Now from Theorem 6, Lemma 3 and the definition of root lattices (Definition 7) it follows that

Theorem 7 *Min($\mathcal{L}$) = Rel($\mathcal{L}$) iff $\mathcal{L}$ is a root lattice.* □

Definition 8

1. *Let $\lambda_0(\mathcal{L}) = 0$, and, for each $r > 0$, let $\lambda_r(\mathcal{L})$ denote the least real number greater than $\lambda_{(r-1)}(\mathcal{L})$ so that there is a vector in $\mathcal{L}$ with norm $\lambda_r(\mathcal{L})$. $\lambda_r(\mathcal{L})$ is called the r-*th norm *in $\mathcal{L}$. (Note: $\mu(\mathcal{L})^2 = \lambda_1(\mathcal{L})$).*

2. *Let $\mathcal{L}_r$ denote the set of all vectors in $\mathcal{L}$ with norm $\lambda_r(\mathcal{L})$. $\mathcal{L}_r$ is called the r-*th shell *in $\mathcal{L}$.*

3. *$\mathcal{L}$ satisfies the* Shell Condition *iff*

   (a) *$\mathcal{L}_r \subset \sum_{i=1}^{r} \mathcal{L}_1$, i.e., every vector in $\mathcal{L}_r$ is the sum of at most r minimal length vectors, and*

(b) $r\lambda_1(\mathcal{L}) \leq \lambda_r(\mathcal{L})$.

Theorem 8 *If $\mathcal{L}$ satisfies the Shell Condition, then Min($\mathcal{L}$) = Rel($\mathcal{L}$).*

PROOF. See Theorem 9, page 474 in [3]. □

Theorem 9 *If $\mathcal{L}$ is minimally generated and Min($\mathcal{L}$) is efficient, then $\mathcal{L}$ satisfies the Shell Condition.*

PROOF.

Suppose $p \in \mathcal{L}$. Then, since $\mathcal{L}$ is minimally generated, there is a shortest Min($\mathcal{L}$)-path $[v_1, \ldots, v_r]$ with endpoint p, so that $\|p\|_{\mathrm{Min}(\mathcal{L})} = r \geq 0$. Then, by Theorem 1, for all $1 \leq i, j \leq r$, $(v_i, v_j) \geq 0$. So, the sequence

$$\|(\sum_{i=1}^{k} v_i)\|, \qquad k = 1, \ldots, r,$$

is strictly increasing and has length r. Therefore, by definition of $\lambda_r(\mathcal{L})$,

$$\|p\|^2 = \|\sum_{i=1}^{r} v_i\|^2 \geq \lambda_r(\mathcal{L}).$$

Also, by Corollary 3,

$$(\forall r \geq 0) \ \left[\frac{\lambda_r(\mathcal{L})}{\mu(\mathcal{L})^2} \text{ is an integer} \right].$$

Clearly, then,

$$(\forall r \geq 0) \ [\lambda_r(\mathcal{L}) \geq r \cdot \mu(\mathcal{L})^2].$$

□

In conclusion, we summarize the above results for minimally generated lattices.

Theorem 10 *The following are equivalent.*

1. *$\mathcal{L}$ is a root lattice.*

2. *$\mathcal{L}$ satisfies the Shell Condition.*

3. *All relevant vectors of $\mathcal{L}$ have equal length.*

4. *$Min(\mathcal{L}) = Rel(\mathcal{L})$.*

5. *$Min(\mathcal{L}) = \overline{Rel(\mathcal{L})}$.*

6. *$\mathcal{L}$ is minimally generated and $Min(\mathcal{L})$ is (economically) efficient.*

7. *$\mathcal{L}$ is minimally generated and every shortest path of length 2 of minimal vectors is intrinsically increasing.*

8. *$\mathcal{L}$ is minimally generated and every shortest path of minimal vectors is intrinsically increasing.*

9. *$\mathcal{L}$ is minimally generated and $Min(\mathcal{L})$ is radial.*

10. *$\mathcal{L}$ is minimally generated and $\| \, \|_{Min(\mathcal{L})}$ is locally compatible with the euclidean norm.* □

7 Discussion

Our motivation for our graphical treatment of lattices comes from certain algorithmic considerations in massively parallel computing—in particular, the simulation of macroscopic physical phenomena in massively parallel computers based on suitable discretizations of euclidean space. Under reasonable assumptions, lattices turn out to be such suitable discretizations of euclidean space (see [2]), and some of the minimally generated lattices are most appropriate (see [2] and [6]). A *lattice computer* model of computation is informally defined as follows. Place synchronized, identical processors at points in a lattice $\mathcal{L}$, and connect two processors by a bi-directional communication channel iff the two corresponding lattice points are $\mu(\mathcal{L})$ apart. Each processor represents a suitable bounded region of euclidean space surrounding the lattice point it is placed on. The lattice computer model is similar to the cellular automaton model, and it has the following features.

1. all the "cells" (processors) are uniform in computational capability as well as *connectivity*, and

2. properties of the underlying lattice can be used in algorithms designed for the model.

The first is clearly desirable from the point of view of designing uniform algorithms for this model; the second turns out to be extremely useful for the domain of applications targeted by this model, i.e., physical phenomena unfolding in euclidean space. In the above approach[2], euclidean space is represented by a suitable lattice computer. One thinks of an object in euclidean space as being represented by the set of lattice computer processors that represent parts of euclidean space occupied by the object. Motion of the object is represented by the passage of messages between connected processors (roughly in the direction of motion) causing appropriate changes in the membership of the set of processors representing the object, over time. In effect the lattice computer as a whole simulates the physical phenomenon under consideration in the discrete universe based on the underlying lattice.

The implicit premise in the above approach is that in a distributed algorithm comprising of local algorithms for simulating individual entities in a physical phenomenon, each local algorithm is simpler, and the distributed algorithm is perhaps more accurate, than one monolithic algorithm for simulating the entire phenomenon—this has been adequately demonstrated by simulations of fluid flow on a Connection Machine (see [5] and [4]). Interestingly enough, although the Connection Machine can be configured to be any general cellular automaton, for the fluid flow simulations the best results were obtained when the machine was configured to be a lattice computer based on A_2, a root lattice.

Algorithms, using the above approach, for simulating two idealized models for physical phenomena are presented in [6]: radiation (at constant radial speed in all directions) from a point source, and the constant speed motion of a particle along a curve. The algorithms are called WAVE$'$ and CURVE, respectively. Both algorithms work in lattice computers of arbitrary dimension. No global routing algorithm is assumed in either, and it is assumed that each processor in the lattice computer can pass messages in one step to any subset of its immediately connected neighbors. Moreover, in both of these algorithms, calculations are based essentially on parameters that are *internal* to the underlying lattice, and thus, exact. Both algorithms produce geometrically "best possible" simulations, with practically no error propagation. Algorithm WAVE$'$ is shown to correctly simulate radiation from a point source in lattice computers based on A_n. The key to the correctness of algorithm WAVE$'$ in this lattice is that the set of minimal length vectors in A_n is what we call a radial set in this paper. Algorithm CURVE is shown to correctly simulate the constant speed motion of a particle along a curve in all those lattice computers based on lattices in which the set of minimal length vectors is equal to the set of relevant vectors—A_n is such a lattice! So both algorithms work correctly in a lattice computer based on A_n. Our aim was to characterize the class of lattice computers in which both these algorithms work correctly. Note that the topology of a lattice computer is just the directed graph (with processors as vertices and connections as edges) induced on the

[2]See [1] for an fuller discussion of this approach.

lattice by the set of minimal length vectors in the lattice. Since our focus is on lattice computers, such a graphical treatment of lattices all but suggests itself.

By the nature of the problem, algorithms for correctly simulating macroscopic physical phenomena in lattice computers must estimate euclidean distances. Also, in the discretized universe of a lattice computer, if objects move by the passing of messages between processors, the graphical path length, measured in terms of the number of connections traversed, is the most easily calculated indicator of distance traversed. Hence it is natural to study the two metrics to see which lattices, if any, allow for any meaningful conversion between the two metrics. Towards this we defined our notion of a graphical path length metric locally compatible with the euclidean metric.

In summary then, the work on lattice computer models of computations led us to study three different kinds of lattices:

1. minimally generated lattices such that the set of minimal length vectors is radial,
2. lattices in which the set of minimal length vectors equals the set of relevant vectors, and
3. minimally generated lattices such that the set of minimal length vectors induces a graphical path length metric that is locally compatible with the euclidean metric.

Surprisingly, as it turns out each of the three constraints above characterizes root lattices!

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