A Characterization of Root Lattices

Dayanand S. Rajan
DCS Inc.
Yardley, PA 19067

Anil M. Shende
(email: shende@bucknell.edu)
Dept. of Computer Science
Bucknell University
Lewisburg, PA 17837

1 Introduction

Every lattice has a set of minimal length vectors and a set of relevant vectors. For lattices in general these two sets are not equal. [1] provides two different sufficient conditions for the set of minimal length vectors in a lattice to be equal to the set of relevant vectors in the lattice. One of these conditions (see the corollary to Theorem 5 on page 459 in [1]) is that the lattice be a root lattice (see Definition 5 below), and the other (see Theorem 9 on page 474 in [1]) is that the lattice satisfy what we call the *Shell Condition*[1] (see Definition 3 below). In this note we show that in fact these two conditions and the equality of the set of minimal length vectors in a lattice and the set of relevant vectors in the lattice are all equivalent, thus providing a characterization of root lattices.

2 Definitions

[1]This name was suggested to us by John Conway and Neil Sloane.

Definition 1 *Suppose $\mathcal{L}$ is an n-dimensional lattice and G is a subset of vectors in $\mathcal{L}$.*

1. *$\| \ \|$ denotes the euclidean norm. $\langle G \rangle$ and $\langle G \rangle_{\mathbf{R}}$ denote the set of linear combinations over the natural numbers and over reals, respectively, of elements of G. A point $p \in \mathcal{L}$ is called a G-neighbor of a point $q \in \mathcal{L}$ iff $p = q + g$ for some $g \in G$. G is a* positive generating set for $\mathcal{L}$ *iff $\langle G \rangle = \mathcal{L}$. G is* weakly acute *iff the inner product of any pair of vectors in G is non-negative. G is* balanced *iff for every $g \in G$, $-g \in G$.* Note: Positive generating sets are never weakly acute, and need not be balanced..

2. *A G-*path *is a finite sequence of elements of G. The empty G-path is denoted $[\ ]$. $[g_1, \ldots, g_k]$, with $g_1, \ldots, g_k \in G$, denotes a G-path. Each element in a G-path is called an* edge *of that G-path. If $P = [g_1, \ldots, g_k]$ is an G-path, and $v \in G$, then $(P \circ v)$ denotes the G-path $[g_1, \ldots, g_k, v]$. The* endpoint of a G-path, *$P = [g_1, \ldots, g_k]$, is the (vector) sum, $\sum_{i=1}^{k} g_i$, of all the edges in P.* Note: We consider the G-path P to begin at the origin of $\mathcal{L}$ and end at the endpoint of P. *The* underlying set *of a G-path P, denoted $\mathbf{U}(P)$, is the set of (pairwise distinct) vectors occurring in P. A G-path is said to be* intrinsically increasing *iff the underlying set of that path is weakly acute. A G-*subpath *of a G-path, P, is a subsequence of consecutively occurring edges in P.*

3. *The* graphical path length *of a G-path, P, is the number of edges in P, and is denoted by $GPL_G(P)$. A G-path, P, is a* shortest G-path *iff every G-path with the same endpoint as that of P has graphical path length at least $GPL_G(P)$.*

For any point $p \in \mathcal{L}$, the graphical distance of p with respect to G, *denoted $||p||_G$, is the path length of a shortest G-path with endpoint p. If there is no G-path to p, then we define $||p||_G$ to be ∞.* Note: $\langle G \rangle = \{p \in \mathcal{L} \mid ||p||_G < \infty\}$.

3 G-Paths and Efficient Sets

Henceforth, $\mathcal{L}$ will always denote a real, n-dimensional lattice, G will denote a subset of vectors in $\mathcal{L}$, and $\vec{0}$ will denote the origin of $\mathcal{L}$. We start this section with the following observations which will be used implicitly through most of this paper.

Lemma 1

1. *G-subpaths of shortest G-paths are shortest G-paths.*
2. *$[g_1, \ldots, g_k]$ is a shortest G-path, iff for every permutation, σ, of degree k, $[g_{\sigma(1)}, \ldots, g_{\sigma(k)}]$ is a shortest G-path in $\mathcal{L}$.*
3. *In any intrinsically increasing path, P, with end-point p, if g is a vector in $\mathbf{U}(P)$, $(p - g, g) \geq 0$.* $\square$

Definition 2 *G is* efficient, *iff the non-zero sum of every pair of vectors in G, whose inner-product is negative, is in G. The* efficient closure *of a set G is the set, denoted by $\overline{G}$, which is the intersection of all efficient supersets of G.*

Then, from definitions and the observations in Lemma 1, it follows that

Theorem 1 *The following are equivalent.*

1. *Every shortest path of vectors from G is intrinsically increasing.*

2. *Every shortest path of length 2 of vectors from G is intrinsically increasing.*

3. *G is efficient.* $\square$

4 Relevant Vectors

In this section, we consider positive generating, efficient $G \subseteq \mathcal{L}$. In Theorem 2 below, we show that the set of relevant vectors of $\mathcal{L}$ (see Definition 3, below) is contained in every such G. We will use the following characterization, due to Voronoi, of relevant vectors as our definition of a relevant vector (see [1, pages 474-5]).

Definition 3 (Voronoi's Characterization) *r is a* relevant vector *of $\mathcal{L}$ iff for every $p \in (\mathcal{L} \setminus \{\vec{0}, r\})$, $(r-2p, r-2p) > (r, r)$. $Rel(\mathcal{L})$ will denote the set of relevant vectors in $\mathcal{L}$.*

We enumerate two restatements of Voronoi's characterization in the corollary below. These are especially useful in our later proofs.

Corollary 1

1. *A point r in $\mathcal{L}$ is a relevant vector iff for every nonzero vector, $p \neq r$, in $\mathcal{L}$, $(p, p) > (p, r)$.*

2. *r is a relevant vector of $\mathcal{L}$ iff $[r]$ is the only intrinsically increasing $\mathcal{L}$-path with endpoint r.* $\square$

A useful geometric interpretation of Corollary 1(1) is that, if r is a relevant vector of $\mathcal{L}$, then, there is no point of $\mathcal{L}$ that lies inside, or on the surface of, the n-dimensional sphere of radius half the length of r and centered at $r/2$.

Theorem 2 *If G is an efficient, positive generating set for $\mathcal{L}$, then G contains $\overline{Rel(\mathcal{L})}$. That is, $\overline{Rel(\mathcal{L})}$ is the minimal efficient positive generating set for $\mathcal{L}$.*

PROOF. Since G is an efficient, positive generating set for $\mathcal{L}$, by Theorem 1, for each relevant vector, r, of $\mathcal{L}$, there is an intrinsically increasing shortest G-path with endpoint r. By Corollary 1(2) this path must be $[r]$, so $r \in G$. Since $\overline{\mathrm{Rel}(\mathcal{L})}$ is the smallest set which is efficient and contains $\mathrm{Rel}(\mathcal{L})$, we have $\overline{\mathrm{Rel}(\mathcal{L})} \subseteq G$. $\square$

Now the following corollary is immediate.

Corollary 2 *If $G \subseteq Rel(\mathcal{L})$ is an efficient, positive generating set for $\mathcal{L}$, then $G = Rel(\mathcal{L}) = \overline{G} = \overline{Rel(\mathcal{L})}$.* $\square$

5 Minimal Vectors and Root Lattices

Every lattice, $\mathcal{L}$, has vectors of a characteristic, to $\mathcal{L}$, minimal length, which we will denote by $\mu(\mathcal{L})$. Also, we will denote the set of all minimal length vectors of $\mathcal{L}$ by $\mathrm{Min}(\mathcal{L})$. Clearly, $\mathrm{Min}(\mathcal{L})$ is a balanced set.

Definition 4 *$\mathcal{L}$ is* minimally generated *iff $Min(\mathcal{L})$ is a (positive) generating set for $\mathcal{L}$.*

Section 2 in Chapter 4 of [1] gives a definition of a root lattice. For our purposes the following, equivalent definition is more useful. (The equivalence follows from Table 4.1 on page 98 in [1] and from the facts that (1) each minimal length vector is a root vector of $\mathcal{L}$, and hence (2) the lattice is generated by the roots of minimal length.)

Definition 5 *$\mathcal{L}$ is a root lattice iff it is minimally generated and the angle between any two distinct minimal length vectors in $\mathcal{L}$ is $k \cdot (\pi/t)$, for some integer k and $t \in \{2, 3\}$.*

All root lattices are minimally generated as are A_n^*, E_6^* and E_7^*. For $n > 4$, D_n^* is not minimally generated. (See [1].)

Theorem 3 *$Min(\mathcal{L}) = Rel(\mathcal{L})$ iff $\mathcal{L}$ is minimally generated and $Min(\mathcal{L})$ is efficient.*

Proof. ($\Longrightarrow$) Since $\mathrm{Min}(\mathcal{L}) = \mathrm{Rel}(\mathcal{L})$, and since $\mathrm{Rel}(\mathcal{L})$ is a balanced positive generating set for $\mathcal{L}$, we need only show that $\mathrm{Min}(\mathcal{L})$ is efficient. Now, let $r_1 \neq r_2$ are relevant vectors so that $(r_1, r_2) < 0$, and let $r = (r_1 + r_2)$. Then,

$$(r, r) < (r_1, r_1) + (r_2, r_2). \tag{1}$$

Suppose, by way of contradiction, r is not relevant. Then, by Corollary 1(1), there exists a $p \in \mathcal{L}$ so that $(p, r) \geq (p, p)$, so that $(p, r - p) \geq 0$. Since r_1, r_2 are of minimal length, $(r, r) \geq (r - p, r - p) + (p, p) \geq (r_1, r_1) + (r_2, r_2)$, contradicting (1) above.

($\Longleftarrow$) Immediate upon replacing $\mathrm{Min}(\mathcal{L})$ for G in Corollary 2. $\square$

Lemma 2 *Suppose $\mathcal{L}$ is minimally generated. Then, Min($\mathcal{L}$) is efficient iff for all $p \neq p' \in$ Min($\mathcal{L}$), $(p,p')/\mu(\mathcal{L})^2 = 0, \pm 1/2$. i.e., iff the angle between any two distinct minimal length vectors in $\mathcal{L}$ is $k \cdot (\pi/t)$, for some integer k and $t \in \{2,3\}$.*

PROOF.

$$\begin{aligned}
&\mathrm{Min}(\mathcal{L}) \text{ is efficient iff}\\
&(\forall p, p' \in \mathrm{Min}(\mathcal{L}) \mid (p,p') < 0)\ (p+p') \in \mathrm{Min}(\mathcal{L}) \text{ iff}\\
&(\forall p, p' \in \mathrm{Min}(\mathcal{L}) \mid (p,p') < 0)\\
&\qquad (p+p', p+p') = (p,p) + 2(p,p') + (p',p') = \mu(\mathcal{L})^2 \text{ iff}\\
&(\forall p, p' \in \mathrm{Min}(\mathcal{L}) \mid (p,p') < 0)\ 2(p,p') = -\mu(\mathcal{L})^2.
\end{aligned}$$

□

Now from Theorem 3, Lemma 2 and the definition of root lattices (Definition 5 it follows that

Corollary 3 *Min($\mathcal{L}$) = Rel($\mathcal{L}$) iff $\mathcal{L}$ is a root lattice.* □

Definition 6

1. *Let $\lambda_0(\mathcal{L}) = 0$, and, for each $r > 0$, let $\lambda_r(\mathcal{L})$ denote the least real number greater than $\lambda_{(r-1)}(\mathcal{L})$ so that there is a vector in $\mathcal{L}$ with norm $\lambda_r(\mathcal{L})$. $\lambda_r(\mathcal{L})$ is called the* r-th norm in $\mathcal{L}$. *(Note: $\mu(\mathcal{L})^2 = \lambda_1(\mathcal{L})$).*

2. *Let $\mathcal{L}_r$ denote the set of all vectors in $\mathcal{L}$ with norm $\lambda_r(\mathcal{L})$. $\mathcal{L}_r$ is called the* r-th shell *in $\mathcal{L}$.*

3. *$\mathcal{L}$ satisfies the* Shell Condition *iff*

   (a) $\mathcal{L}_r \subset \sum_{i=1}^r \mathcal{L}_1$, *i.e., every vector in $\mathcal{L}_r$ is the sum of at most r minimal length vectors, and*

   (b) $r\lambda_1(\mathcal{L}) \leq \lambda_r(\mathcal{L})$.

Theorem 4 (Theorem 9, page 474 in [1]) *If $\mathcal{L}$ satisfies the Shell Condition, then $Min(\mathcal{L}) = Rel(\mathcal{L})$.* □

Theorem 5 *If $\mathcal{L}$ is minimally generated and $Min(\mathcal{L})$ is efficient, then $\mathcal{L}$ satisfies the Shell Condition.*

PROOF.

Suppose $p \in \mathcal{L}$. Then, since $\mathcal{L}$ is minimally generated, there is a shortest $\mathrm{Min}(\mathcal{L})$-path $[v_1, \ldots, v_r]$ with endpoint p, so that $||p||_{\mathrm{Min}(\mathcal{L})} = r \geq 0$. Then, by Theorem 1, for all $1 \leq i, j \leq r$, $(v_i, v_j) \geq 0$. So, the sequence

$$||(\sum_{i=1}^{k} v_i)||, \qquad k = 1, \ldots, r,$$

is strictly increasing and has length r. Therefore, by definition of $\lambda_r(\mathcal{L})$,

$$||p||^2 = ||\sum_{i=1}^{r} v_i||^2 \geq \lambda_r(\mathcal{L}).$$

Since $\mathcal{L}$ is minimally generated, each point q in $\mathcal{L}$ is the endpoint of a Min($\mathcal{L}$)-path, $[v_1, \ldots, v_k]$. Then,

$$(q, q) = \left[\sum_{1 \le i \le k} (v_i, v_i) \right] + 2 \left[\sum_{1 \le i < j \le k} (v_i, v_j) \right],$$

which, by Lemma 2, equals $\mu(\mathcal{L})^2 \cdot t$, for some integer t. Therefore,

$$(\forall r \ge 0) \ \left[\frac{\lambda_r(\mathcal{L})}{\mu(\mathcal{L})^2} \text{ is an integer} \right].$$

Clearly, then,

$$(\forall r \ge 0) \ [\lambda_r(\mathcal{L}) \ge r \cdot \mu(\mathcal{L})^2].$$

$\square$

In conclusion, we summarize the above results.

Theorem 6 *The following are equivalent.*

1. *$\mathcal{L}$ is a root lattice.*
2. *$\mathcal{L}$ satisfies the Shell Condition.*
3. *Min($\mathcal{L}$) = Rel($\mathcal{L}$).* $\square$

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References

[1] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups.* Springer Verlag, 1993. Second edition.